# On the Complexity of Language Membership for Probabilistic Words

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## Introduction and Example

We consider **probabilistic words**:

- Probabilistic word $p$ of length $n$ over $\Sigma = \{a, b\}$: sequence $p = p_1 \cdots p_n$ of probability distributions over letters of $\Sigma$.
  - $\rightarrow$ For instance, for all $1 \leq i \leq n$, set $p_i(a) := 0.6$ and $p_i(b) := 0.4$
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Ex: for $p$ above and $L_0$, we have $p(L_0) = \sum_{0 \leq j \leq \lfloor n/2 \rfloor} \binom{n}{2j} 0.6^{2j} 0.4^{n-2j}$

# Probabilistic Membership Problem

For any fixed language $L$, define the **probabilistic membership problem** $\#\mathrm{pM}(L)$:

**Problem #pM($L$)**

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- **Naïve algorithm** (exponential):
  - Consider every possible word $w = a_1 \cdots a_n$ of $\Sigma^n$
  - Compute its probability $p(w) := \prod_{1 \leq i \leq n} p_i(a_i)$
  - Test whether $w \in L$
  - Sum the $p(w)$ for all such words

→ **Research question**: For which languages $L$ is $\#\text{pM}(L)$ in PTIME?

# Roadmap

Tractability for Regular Languages and Unambiguous CFLs

Intractability for Some Inherently Ambiguous Context-Free Languages

Other Methods for Tractability

# Tractability for Regular Languages and Unambiguous CFLs

# Tractability for Regular Languages

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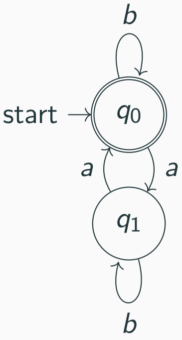


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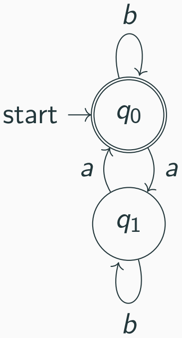


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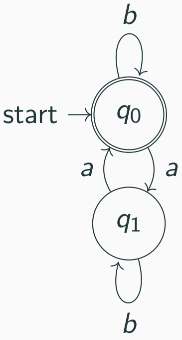


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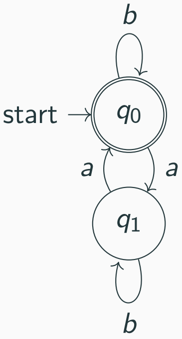


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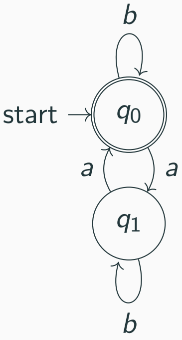


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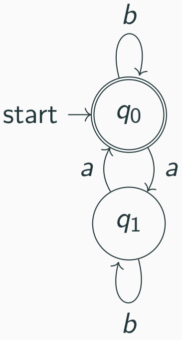


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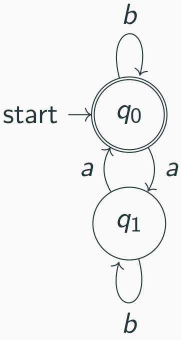


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This even works for unambiguous automata (only one accepting run per word)

# Tractability for UCFG (Result Statement)

We study #pM($L$) for **context-free languages** (CFL) $L$. Recall the definitions:

- A CFL is a language recognized by a **context-free grammar** (CFG):
  - Ex: $L_1 = \{a^n b^n \mid n \in \mathbb{N}\}$ with the CFG $G_1$ having productions $S \to aSb$ and $S \to \varepsilon$
- A word is accepted by $G$ if it has a **derivation tree**
- A CFG is **unambiguous** (UCFG) if every word has at most one derivation tree
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  - Example: $L_1$ is a UCFL

**Theorem**

For any UCFL $L$, the problem #pM($L$) is in PTIME

# Tractability for UCFG (Proof Sketch)

- Recall the **CYK** algorithm to test if a word $w = a_1 \cdots a_n$ is accepted by a CFG $G$
  - Build a table of Booleans $T[A, i, j]$ indicating whether $a_i \cdots a_{j-1}$ is derivable from $A$
  - Inductive case: $T[A, i, j] = \bigvee_{A \to BC} \bigvee_{i \leq k \leq j} T[B, i, k] \wedge T[C, k, j]$

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- Generalization on a probabilistic word $p = p_1 \cdots p_n$: build a table of rationals $P[A, i, j]$ indicating the probability that $p_i \cdots p_{j-1}$ generates a word derivable from $A$
- Inductive case: $P[A, i, j] = \sum_{A \to BC} \sum_{i \leq k \leq j} P[B, i, k] \times P[C, k, j]$
- Uses **independent AND** (disjoint positions) and **exclusive OR** (unambiguity)

# Intractability for Some Inherently Ambiguous Context-Free Languages

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- The input words will be of the form: $p = u \ \$ \ v_1^R \ \$ \ \cdots \ \$ \ v_m^R \ \$$ where
  - $u$ is a uniform probabilistic word coding a valuation
  - $v_i^R$ is (the mirror of) a non-probabilistic word coding the $i$-th clause

# Hardness for a Fixed Grammar (II)

Example: $\phi = (x_1 \wedge x_2) \vee (x_2 \wedge x_3)$

Input words of the form $p = u \,\$\, v_1^R \,\$\, v_2^R \,\$$ with:

- $u = u_1 u_2 u_3$ with $u_i(0) = u_i(1) = 1/2$ for each $i$
- $v_1 = 110$ and $v_2 = 011$
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Thus: #pM($L_0$) is #P-hard for the language $L_0$ over $\Sigma = \{\$, 0, 1\}$ defined as:

$$L_0 := \{\ u \,\$\, \Sigma^* \,\$\, v^R \,\$\, \Sigma^* \mid u, v \in \{0,1\}^* \text{ with } v \leq_{\text{componentwise}} u\ \}$$

$L_0$ can be recognized by a nondeterministic pushdown automaton

# Other Hard Languages

1. **The union of 2 UCFGs can be #P-hard**
   - Encode the run of a polynomial-time nondeterministic Turing Machine:
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$\Rightarrow$ Is #pM($L$) hard for all inherently ambiguous CFLs $L$?

# Other Methods for Tractability

# Poly-Slicewise-Unambiguous Languages

$L$ is **poly-slicewise-unambiguous** if, for each length $n$, we can compute in PTIME a UCFG capturing $L \cap \Sigma^n$

**Theorem**

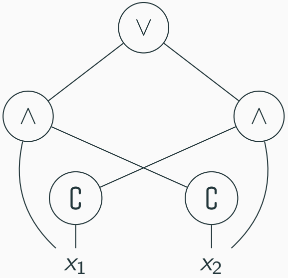
*If $L$ is poly-slicewise-unambiguous, then $\#\mathrm{pM}(L)$ is in PTIME*

**Consequences:**

- #pM is tractable for **polyslender / bounded CFLs**
  - Polyslender = number of words of length $n$ is polynomial in $n$
  - The UCFG for length $n$ just generates each word explicitly
  - Covers some inherently ambiguous CFLs, e.g., $\{a^i b^j c^k \mid i = j \vee j = k\}$
- #pM is tractable for **unambiguous polynomial-time counter automata**
  - In particular: VASS, Parikh automata, one-counter automata, etc.
  - By contrast nondeterministic counter automata may be hard (previous slide)

# Tractable Circuits

**Boolean circuit** $C$ over variables $x_1, \ldots, x_n$: directed acyclic graph (DAG) with $\wedge$ / $\vee$ / $\neg$ gates (AND, OR, negation).



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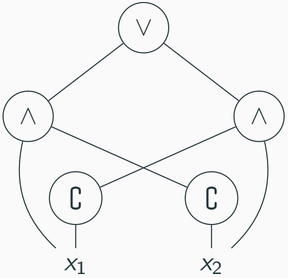
$$L(C) = \{\, w \in \{0,1\}^n \mid C(w) = 1 \,\}.$$



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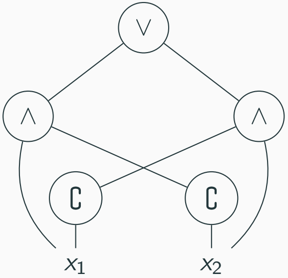
$$L(C) = \{\, w \in \{0,1\}^n \mid C(w) = 1 \,\}.$$

$\#$pM($L(C)$) then amounts to computing the probability that $C$ evaluates to true (but this may be $\#$P-hard)



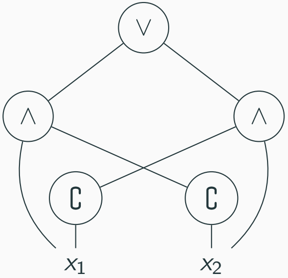
# Tractability of Circuits

On which circuits $C$ can we tractably compute the probability that $C$ is true?

**Tractable circuits**

A circuit is **tractable** when:

- $\land$ gates are **decomposable**: no common variable in both inputs
- $\lor$ gates are **disjoint**: no common satisfying assignments



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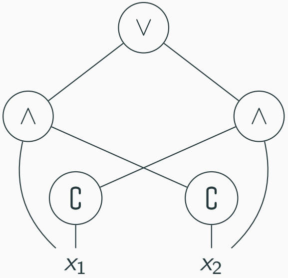
**Theorem**

*Given a tractable circuit $C$ on inputs $x_1, \ldots, x_n$ and a probabilistic word $p = (p_1, \ldots, p_n)$, we can solve $\#pM(L(C))$ in linear time (up to arithmetic costs)*



# Grammars vs Circuits

- Given a CFL $L$ recognized by CFG $G$ and length $n$ we can build in PTIME a circuit $C_n$ with $L(C^n) = L \cap \Sigma^n$
  - Proof: Trace of the CYK algorithm
- If the grammar is unambiguous, then we get a tractable circuit
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- The converse is known to be false: $\{\Sigma^k a \Sigma^{n-1} a \Sigma^{n-1-k} \mid k \leq n-1\}$
  - $\rightarrow$ has tractable circuits, but not poly-slicewise-unambiguous
  - $\rightarrow$ see [Mengel and Vinall-Smeeth, PODS'25]

$\Rightarrow$ Intuition: tractable circuits allow an arbitrary reading order (vs left-to-right order for poly-slicewise-unambiguous)

## Application to $\mathrm{PAL}^2$

$\mathrm{PAL}^2$ is the set of words of the form $w = w_1 w_2$ where $w_1$ and $w_2$ are palindromes

**Theorem**

*$\#pM(\mathrm{PAL}^2)$ can be solved in $O(n^3)$ by building tractable circuits*

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**Proof idea**

- Key combinatorial lemma: The primitive root of any word in $\mathrm{PAL}^2$ decomposes uniquely into two palindromes
- Then do a downward induction based on the **order** of the word
- Use subset difference to do inclusion-exclusion reasoning
- Same technique can show tractability of $\#$pM for the language of primitive words

# Conclusion and Future Work

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- We studied probabilistic membership $\#\mathrm{pM}(L)$ for CFLs
- The problem is tractable for:
  - Unambiguous CFLs
  - Poly-slicewise-unambiguous languages, including polyslender languages and unambiguous polynomial-time counter automata
  - Languages with tractable $\times, \uplus, \complement$-circuits, including primitive words and $\mathrm{PAL}^2$
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## Future directions

- Classify more CFLs (e.g., $\mathrm{PAL}^k$, $\mathrm{PAL}^*$, etc.)
- Are tractable circuits with negation more powerful than without?
- Study combined complexity, with the automaton/CFG also given as input
- Complexity of uniform $\#\mathrm{pM}$? (all positions have the same distribution)
- Explore links with classical parsing complexity

# Conclusion and Future Work

## Conclusion

- We studied probabilistic membership $\#\mathrm{pM}(L)$ for CFLs
- The problem is tractable for:
  - Unambiguous CFLs
  - Poly-slicewise-unambiguous languages, including polyslender languages and unambiguous polynomial-time counter automata
  - Languages with tractable $\times, \uplus, \complement$-circuits, including primitive words and $\mathrm{PAL}^2$
- Intractable already for:
  - Some unions of two UCFLs
  - Some nondeterministic counter automata

## Future directions

- Classify more CFLs (e.g., $\mathrm{PAL}^k$, $\mathrm{PAL}^*$, etc.)
- Are tractable circuits with negation more powerful than without?
- Study combined complexity, with the automaton/CFG also given as input
- Complexity of uniform $\#\mathrm{pM}$? (all positions have the same distribution)
- Explore links with classical parsing complexity

**Thanks for your attention!**

# Tractability for Primitive Words (Result)

**Definition**

A word $w \in \Sigma^*$ is *primitive* if it is not a proper power:

$$w \neq u^k \quad \text{for any } u \in \Sigma^*,\ k \geq 2.$$

Let $L_{\mathrm{prim}}$ be the language of primitive words.

**Theorem**

*The problem* $\#\mathrm{pM}(L_{\mathrm{prim}})$ *can be solved in time* $O(n^2)$

Note that it is not known whether $L_{\mathrm{prim}}$ is context-free

# Tractability for Primitive Words (Proof Sketch)

**Root and order**

Every $u \neq \varepsilon$ can be uniquely written as $u = w^d$ where $w$ is primitive (the root) and $d \geq 1$ is the order.

Define:

$$L_k = \{\text{words of order } k\}, \qquad M_k = \{v^k \mid v \in \Sigma^*\}.$$

Note that $M_k$ is the set of words whose order is a multiple of $k$.

**Step 1:** Build circuits for $M_k$ (with $k \mid n$)

- Enforce periodicity: test that we have the same letter at each offset
- Product over offsets, union over letters, product over positions having that offset

**Step 2:** Recover $L_1 = L_{\mathrm{prim}}$ by

$$M_k = \biguplus_{d \geq 1} L_{dk}$$

Downward induction, and subset difference via complementation: $V \setminus U = \complement((\complement V) \uplus U)$