Regular Languages: Some New Problems and Algorithms

Antoine Amarilli

November 26, 2025

Joint work with: Corentin Barloy, Pierre Bourhis, İsmail İlkan Ceylan, Sven Dziadek, Octave Gaspard, Wolfgang Gatterbauer, Paweł Gawrychowski, Benoît Groz, Santiago Guzman Pro, Louis Jachiet, Sébastien Labbé, Neha Makhija, Kuldeep Meel, Stefan Mengel, Mikaël Monet, Martín Muñoz, Matthias Niewerth, Charles Paperman, Paul Raphaël, Tina Ringleb, Cristian Riveros, Sylvain Salvati, Luc Segoufin, Tim Van Bremen, Nicole Wein

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  $\rightarrow$ determine if data satisfies a query
- Today’s talk: regular language membership

  $\rightarrow$ determine if a word belongs to a regular language

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- Boycotting reviewing and involvement in conferences with mandatory in-person attendance
  → both an inclusivity problem and a sustainability problem
- TCS4F: measure and limit the carbon footprint of TCS research

TCS4F
THEORETICAL
COMPUTER
SCIENTISTS
FOR FUTURE

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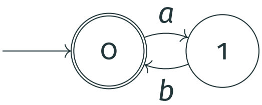
Open access: TCS research is published behind paywalls and/or with high article processing charges, because of parasitic publishers

- Since 2016: not reviewing for closed-access conferences
- NFVNR: encourage other researchers to do the same

Membership to regular languages

Regular languages

Regular languages are a robust framework for constant-memory computation:



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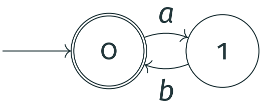

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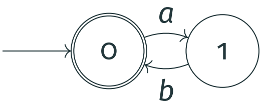

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The computational complexity can be studied in two settings:

- Data complexity: the language L is fixed and the input is w
- Combined complexity: the input is both w and some representation of L

Membership problem

What is the complexity of membership?

- In data complexity: can be decided in $O(|w|)$ on an input word w
  - $O(1)$ possible for some languages [Aaronson et al., 2019]

$\rightarrow$ We can check the word length modulo p in $O(1)$

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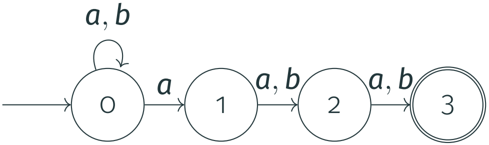

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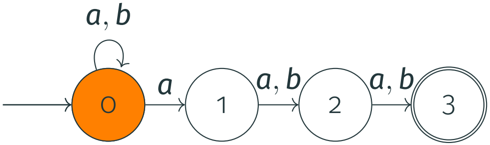

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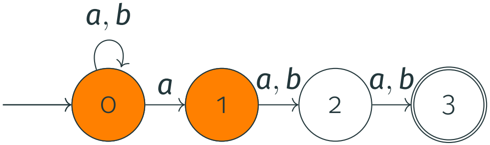

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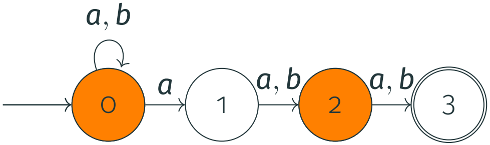

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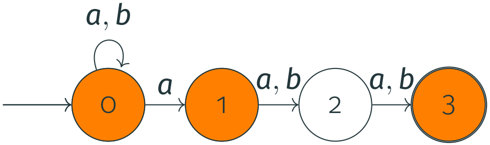

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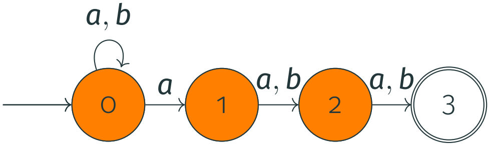

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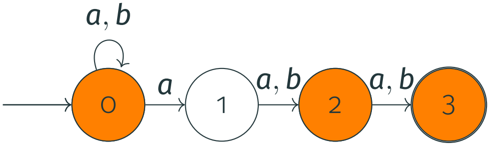

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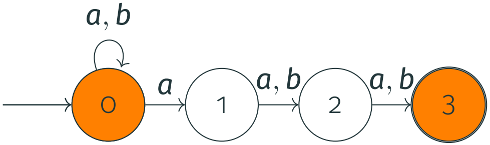

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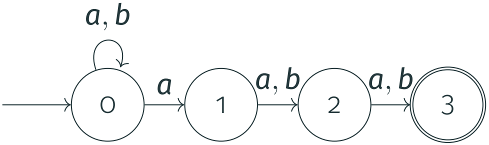

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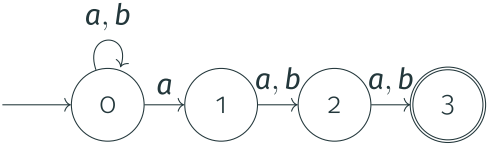

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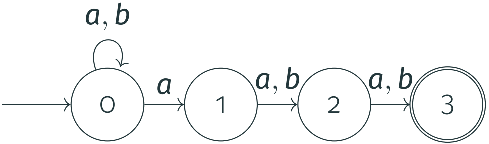

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In this talk we will look at extensions of the membership problem

Roadmap

I will present **four extensions** of regular language membership...

- Membership for **partial words** and **probabilistic words**
- **Incremental maintenance** of membership
- **Enumeration** of word factors (beyond Boolean queries)
- Regular language problems on **graphs**

... and will sketch more directions at the end.

Partial and Probabilistic Words

Partial and probabilistic words

- Partial word: word with holes
  - → e.g., $a\_b\_$ on alphabet $\Sigma = \{a, b\}$
- Possible completions: filling the holes with letters
  - → here, 4 possible completions
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  Given a partial word w, how many completions of w belong to L?
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- This defines a **probability distribution** on possible completions
- **Membership for probabilistic words** to L: Given a probabilistic word w, what is the **total probability** of the completions of w that belong to L?

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Data complexity (fixed regular language L): in linear time (up to arithmetic costs)

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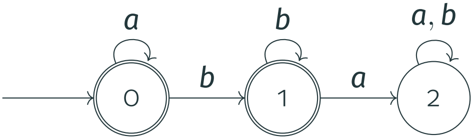

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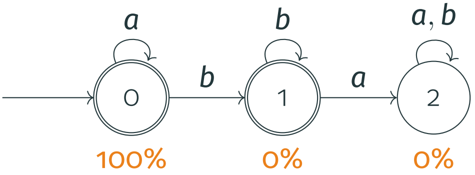

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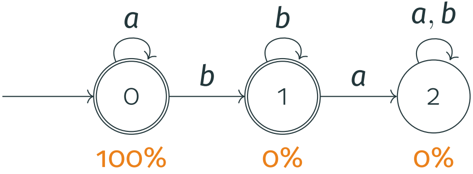

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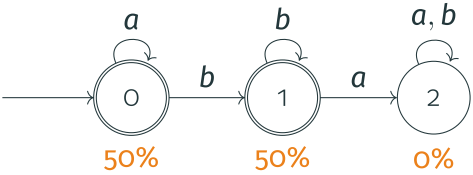

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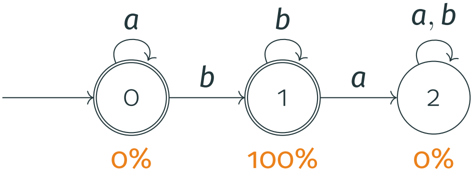

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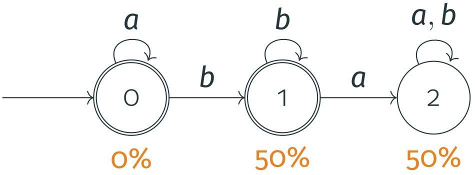

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In combined complexity:

- It is #P-hard in general but can be approximated (FPRAS) [Arenas et al., 2021]
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Generalizations to context-free grammars: [A., Monet, Raphaël, Salvati, 2025]

- It is in PTIME for unambiguous CFGs
- It is #P-hard already for some 2-ambiguous linear CFGs
- Other tractable cases:
  - Polyslender CFGs, e.g., $\{a^n b^n \mid n \in \mathbb{N}\}$
  - Some unambiguous counter automata
  - Some ad-hoc languages, e.g., concatenations of two palindromes

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Many different kinds of updates:

- Endpoint updates (next slide)
- Substitution updates (main focus)
- Other updates (insert/delete, cut and paste, etc.)

Dynamic membership under endpoint updates

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  - Not obvious (relates to simulating a queue with stacks), cf [Ganardi et al., 2022]
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- Push-right and pop-right: add/remove letters at the end of the word (stack)
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OK, what about updates inside the word?

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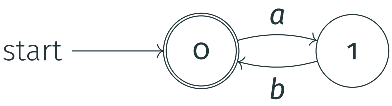
Theorem

For any regular language L recognized by an NFA A, given a word w, we can maintain dynamic membership of w to L under substitution updates in $O(Poly(|A|) \times \log |w|)$ combined complexity per update

Proof sketch of the $O(\log n)$ algorithm

Fix the language $L = (ab)^*$:



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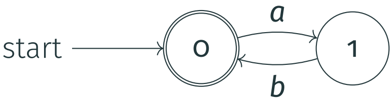

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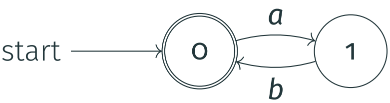

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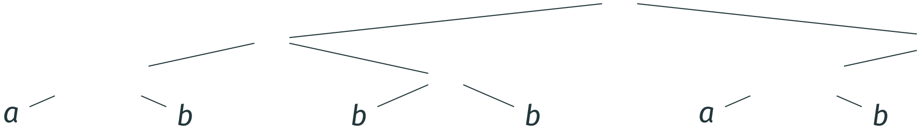

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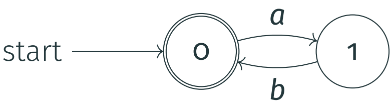

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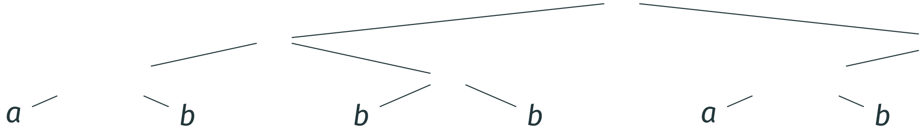

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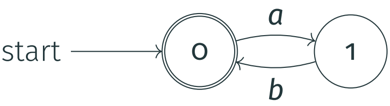

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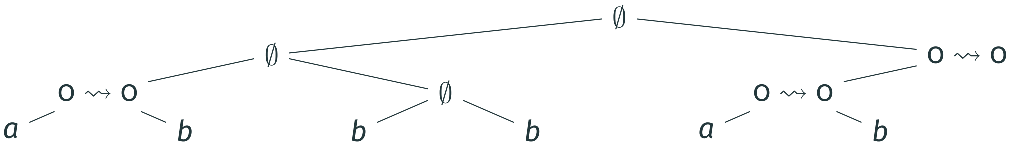

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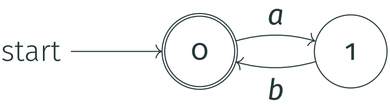

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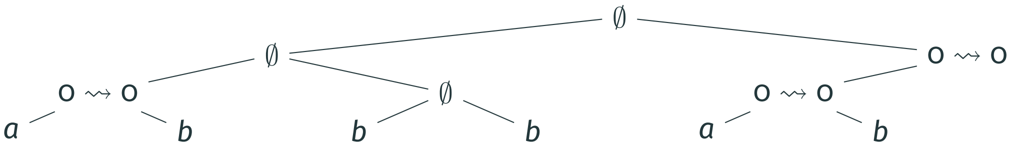

- The **tree root** describes if $w \in L$
- We can update the tree for each substitution **in $O(\log n)$**
- Can be improved to **$O(\log n / \log \log n)$**

Improving on $O(\log n)$ for some languages

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Question:

What is the data complexity of dynamic membership, depending on the fixed regular language L?

Summary of our results [A., Jachiet, Paperman, 2021]

QLZG: in $O(1)$

QSG: in $O(\log \log n)$
not in $O(1)$?

All: in $\Theta(\log n / \log \log n)$

- We identify a class **QLZG** of regular languages:
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Generalizations to trees: [A., Barloy, Jachiet, Paperman, 2025] and Labbé's PhD

Enumeration of Factors

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Factor enumeration problem:

- **Fix:** regular language L
- **Input:** word $w = a_1 \cdots a_n$
- **Output:** enumerate all pairs $[i, j\rangle$ such that $a_i \cdots a_{j-1} \in L$

$\rightarrow$ This is like `re.findall` but returning all pairs (including overlapping ones)

Measuring the complexity

- Naive algorithm: Run a DFA A for L on each factor of w

```
l o l
```

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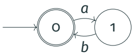
→ We need to measure complexity differently

Enumeration algorithms

```
Antoine Amarilli Description Name Antoine
Amarilli.  Handle:  a3nm.  Identity Born
1990-02-07.  French national.  Appearance as
of 2017.  Auth OpenPGP. OpenId.  Bitcoin.
Contact Email and XMPP a3nm@a3nm.net
Affiliation Associate professor ...
```

Word w



Automaton A

Enumeration algorithms

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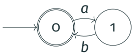
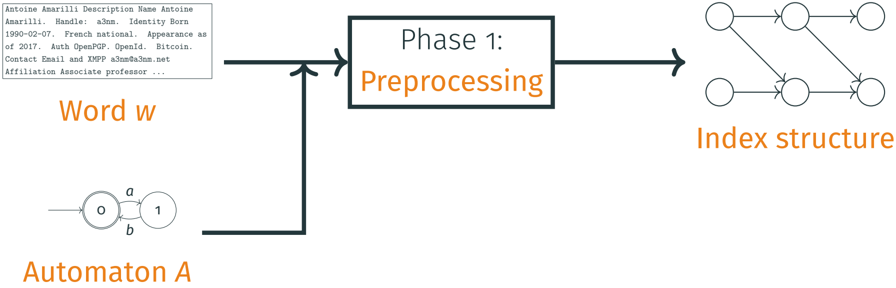


Automaton A

Phase 1:

Preprocessing

Enumeration algorithms



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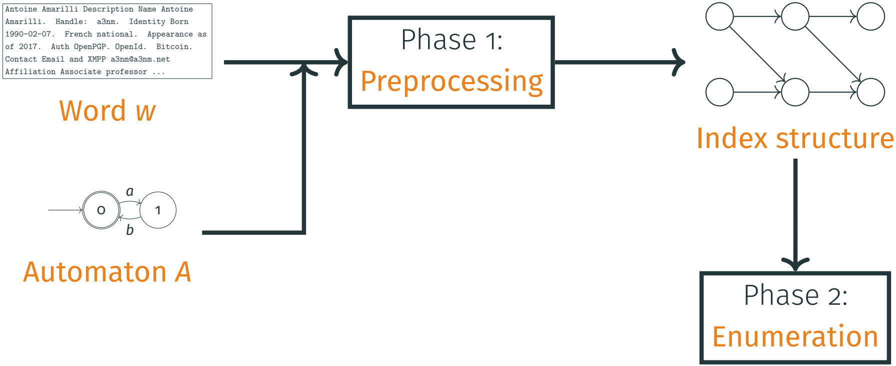

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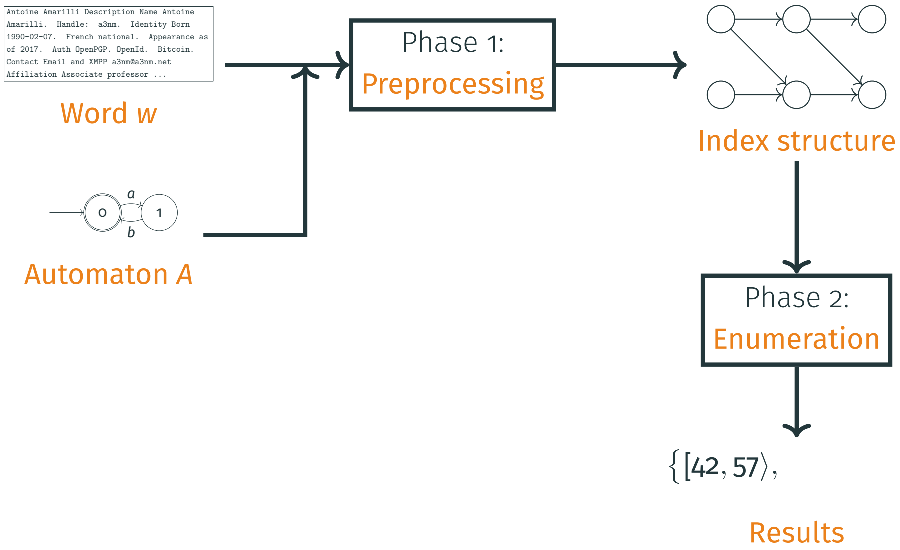

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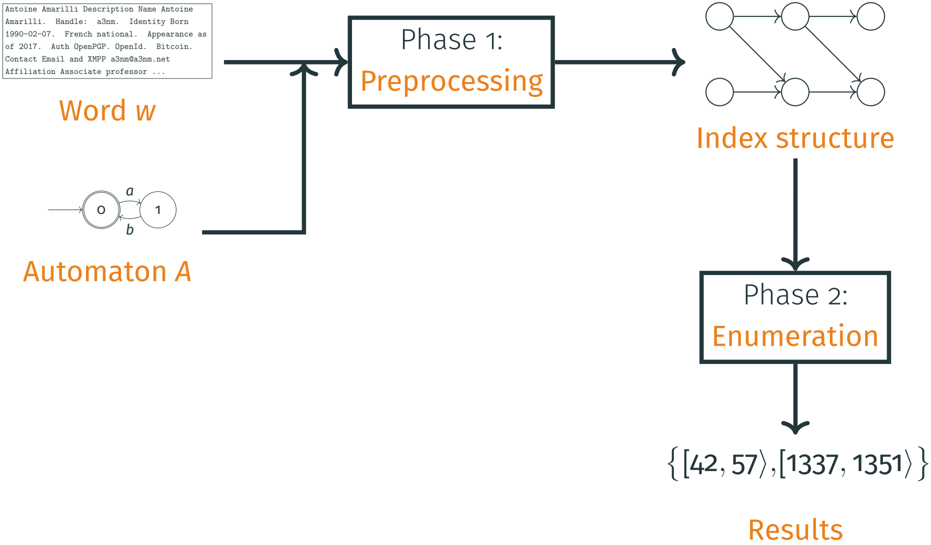

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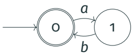

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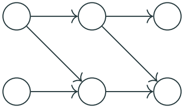

Index structure

Phase 2: Enumeration

$\{[42, 57\rangle, [1337, 1351\rangle\}$

Results

Two ways to measure performance:

- Total time for phase 1
- Delay between two results in phase 2

... as a function of the word and automaton

Results for factor enumeration

Existing work has shown the best possible bounds :

Theorem (follows from [Florenzano et al., 2018])

Given a word w and a DFA A, we can enumerate the factors in w that match A with preprocessing $O(|w| \times |A|)$ and delay $O(1)$.

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Theorem ([A., Bourhis, Mengel, Niewerth, 2019a])

For a NFA A, we can enumerate the factors of w that match A with preprocessing $O(|w| \times Poly(|A|))$ and delay $O(Poly(|A|))$.

Automata with captures

The result extends beyond factor enumeration to automata with captures

$\rightarrow$ Example: $(a|b)^* \ \alpha \ a^* \ \beta \ (a|b)^*$

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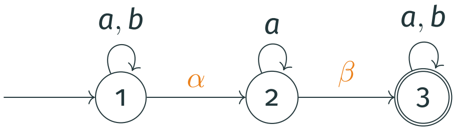

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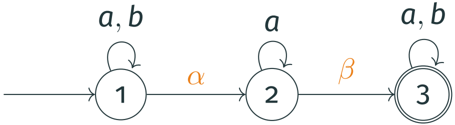

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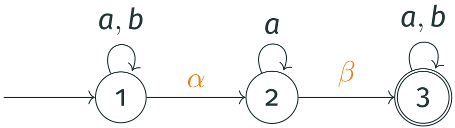

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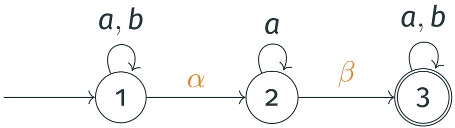

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Also: generalizations to tree automata [A., Bourhis, Mengel, Niewerth, 2019b] and context-free grammars: [A., Jachiet, Muñoz, Riveros, 2022]

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Compute a product DAG of the word W and of the automaton A

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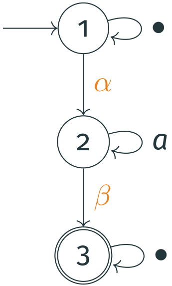

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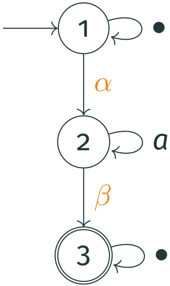

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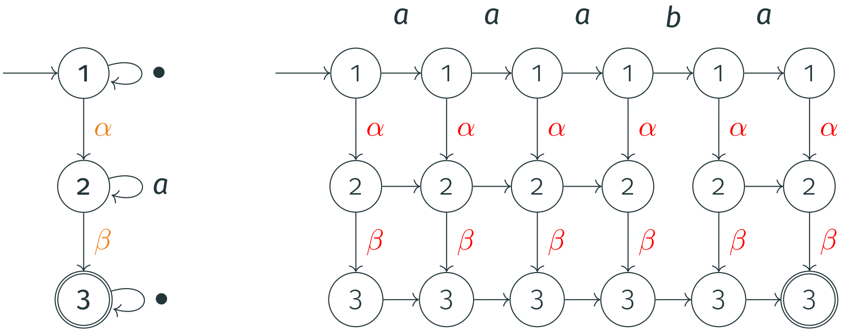

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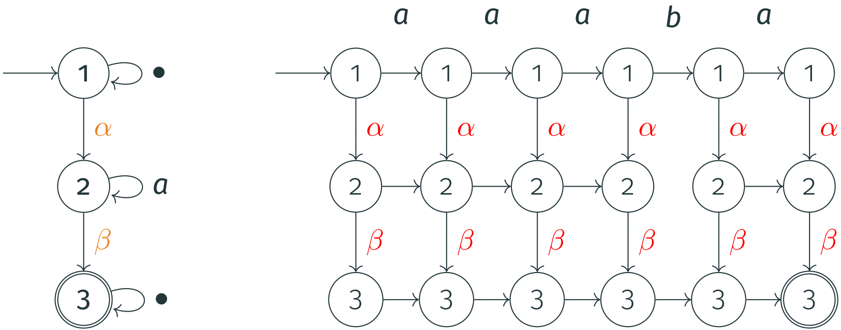

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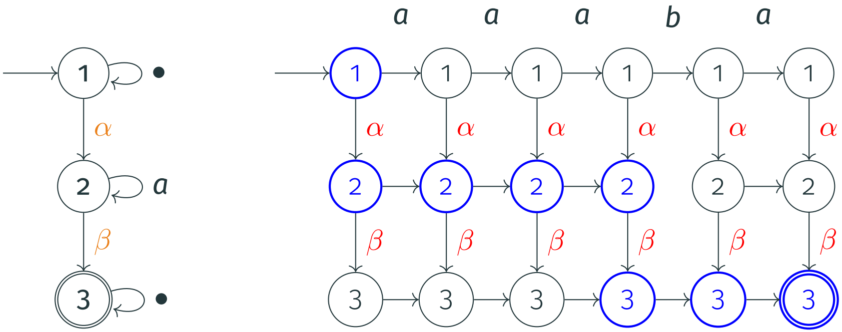

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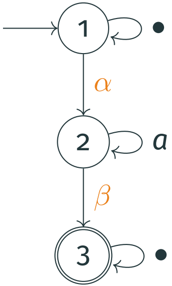

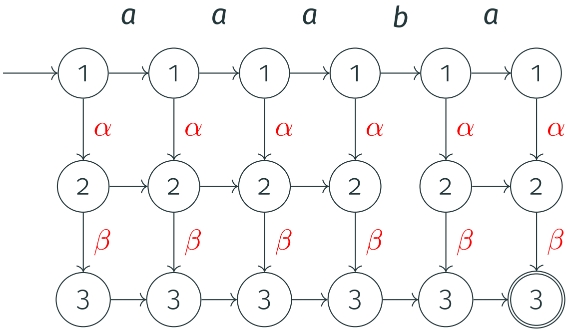

$\rightarrow$ Each path in the product DAG corresponds to a match

Challenge: Enumerate paths but avoid duplicate matches

Regular Languages on Graphs

Regular Path Queries (RPQs)

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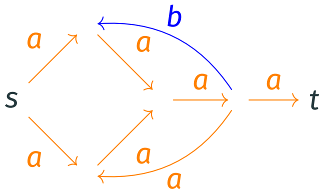

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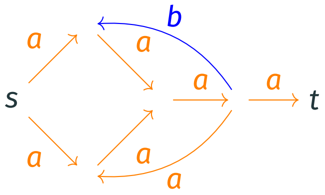

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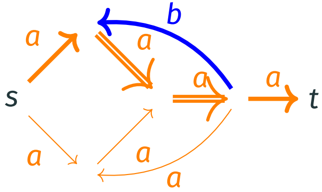

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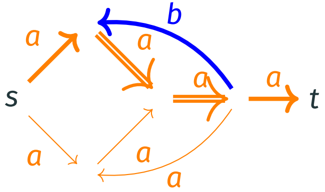

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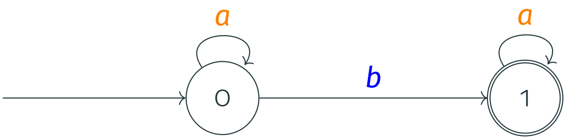

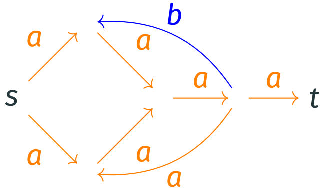

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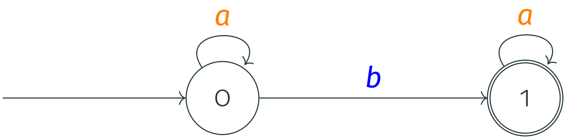

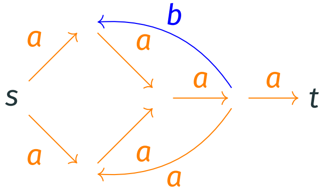

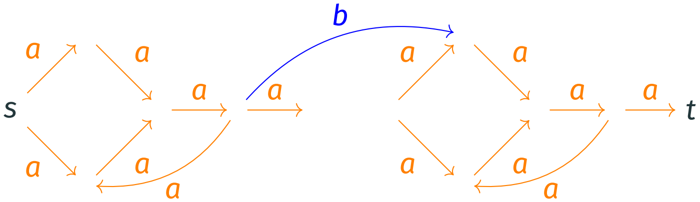

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We have defined the RPQ evaluation problem, which generalizes regular language membership to graph databases:

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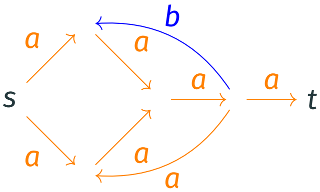

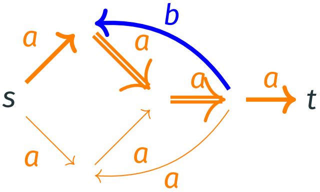

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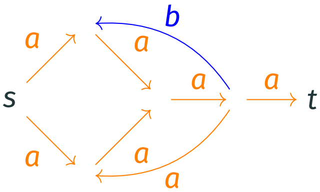

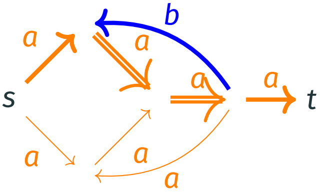

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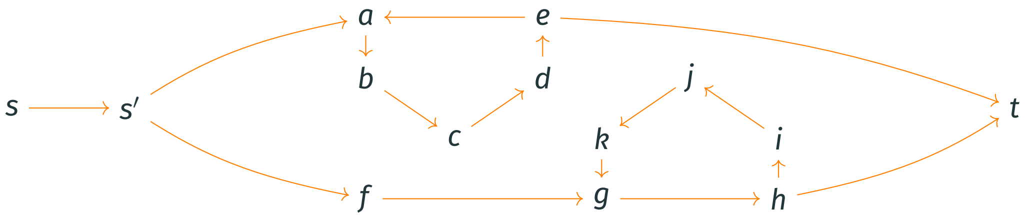

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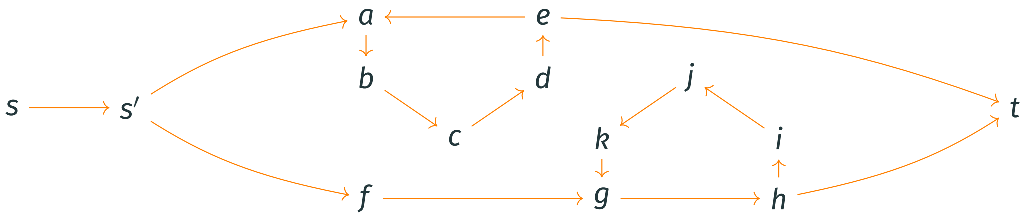

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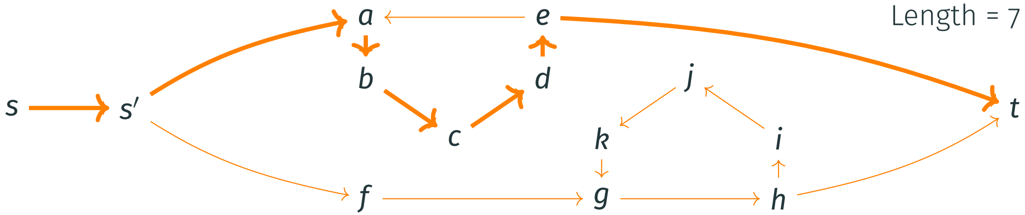

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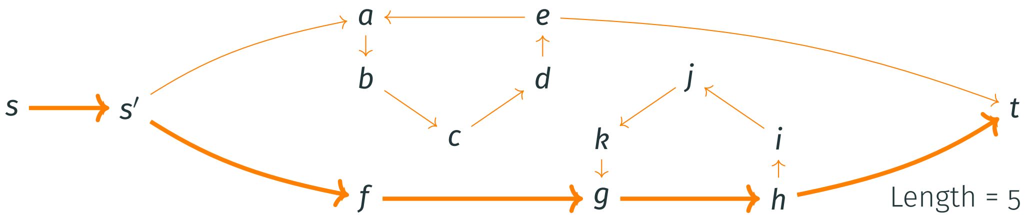

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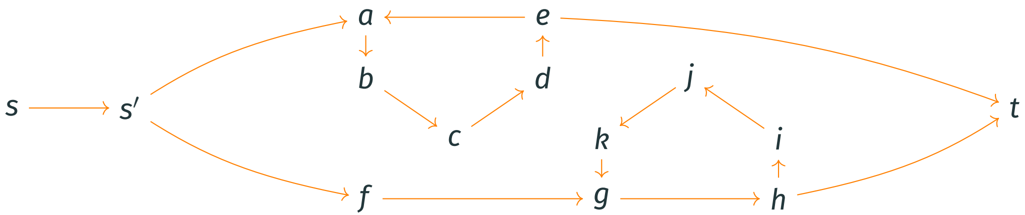

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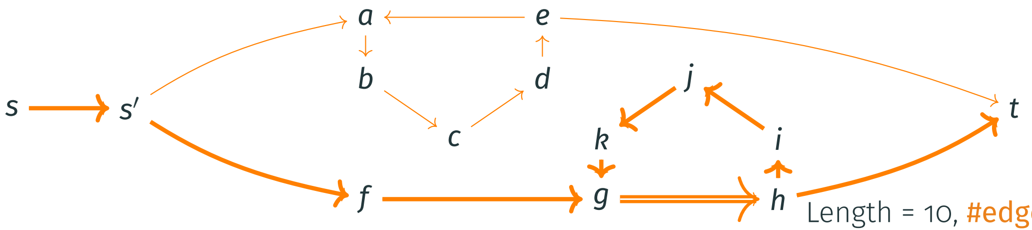

Length = 10, #edges = 9

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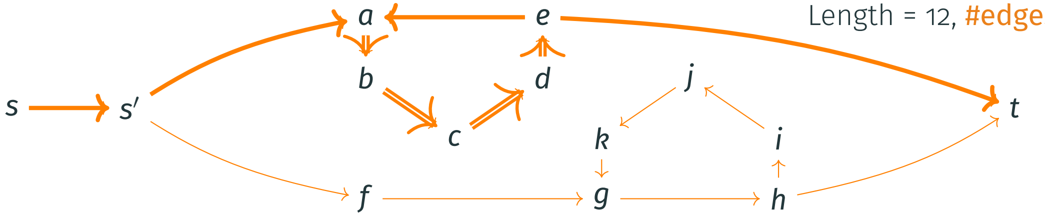

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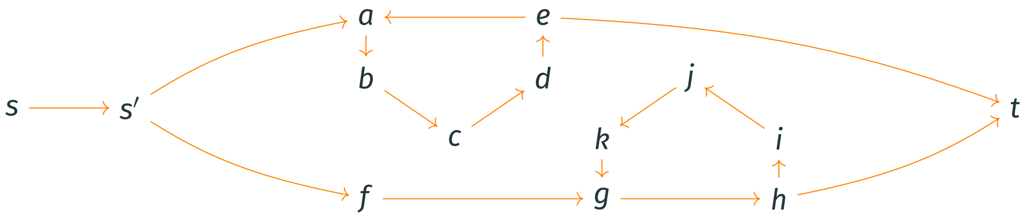

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- Is this problem in PTIME?

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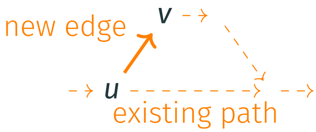

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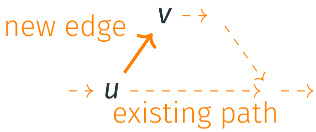

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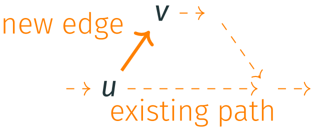

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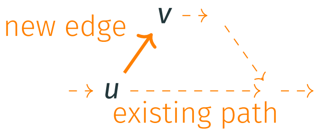

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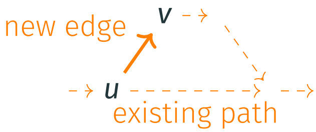

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- Bounded-cutwidth subgraphs can be found by **dynamic programming**

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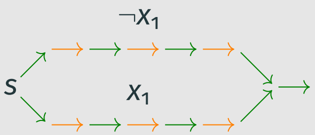

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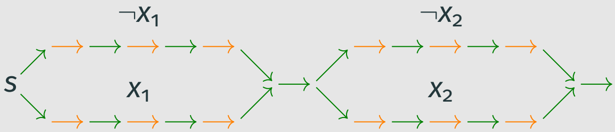

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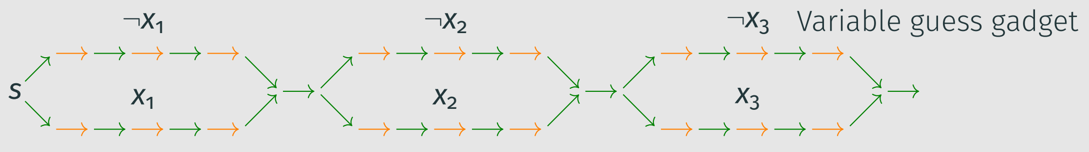

t

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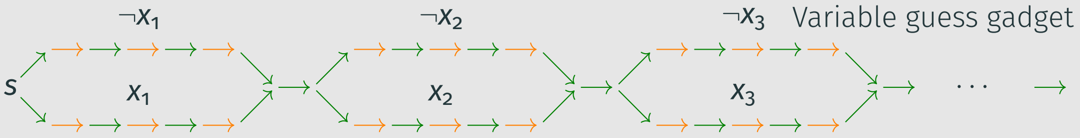

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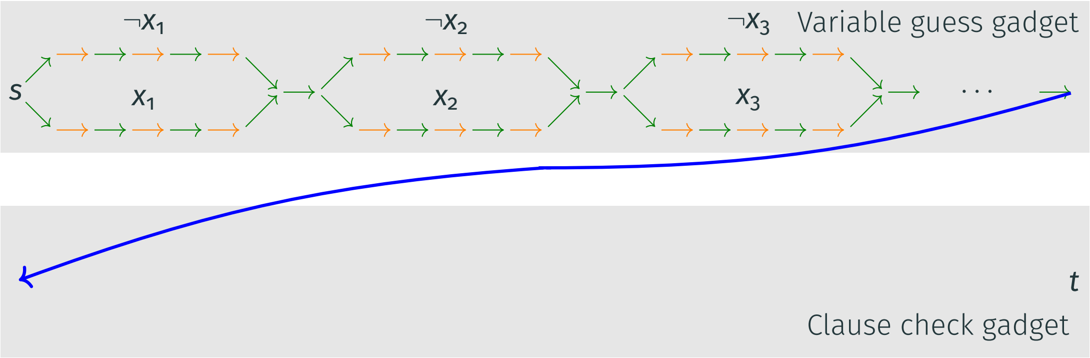

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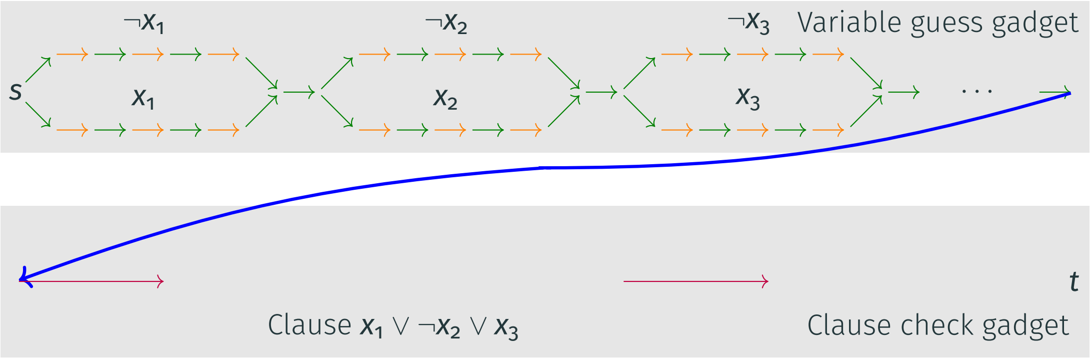

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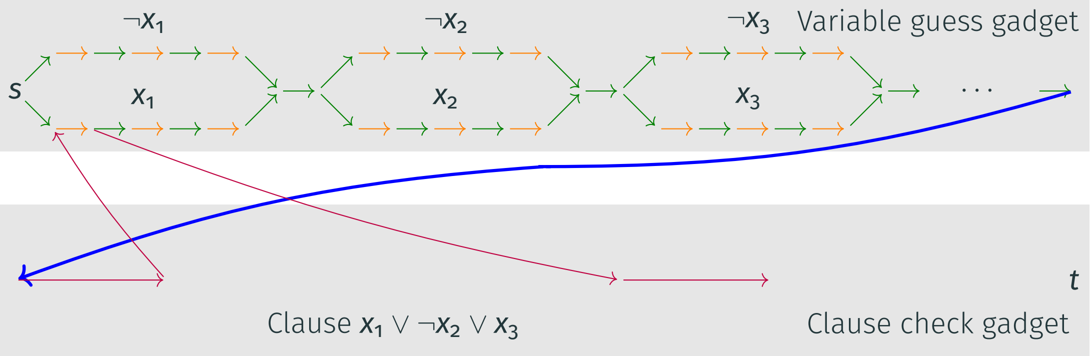

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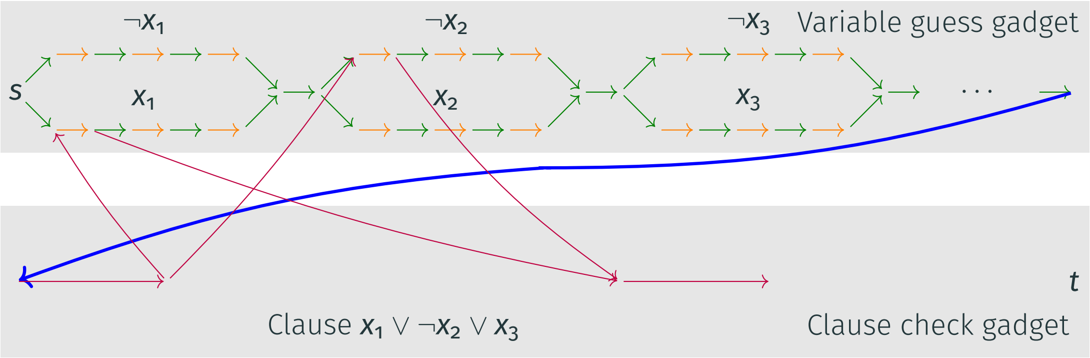

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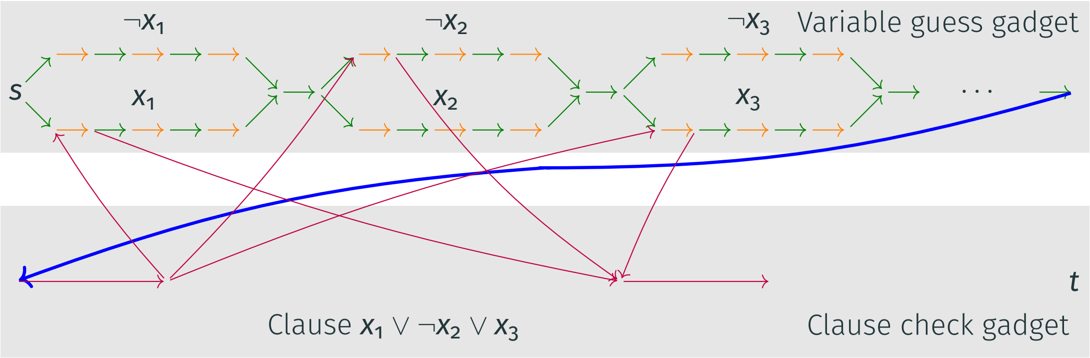

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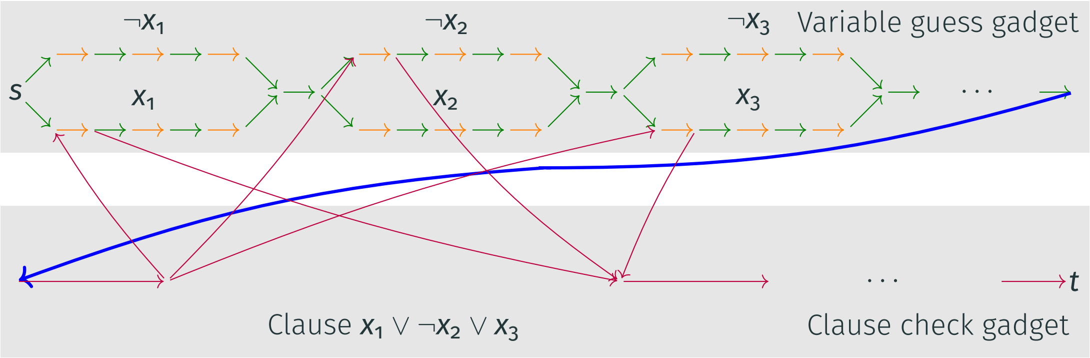

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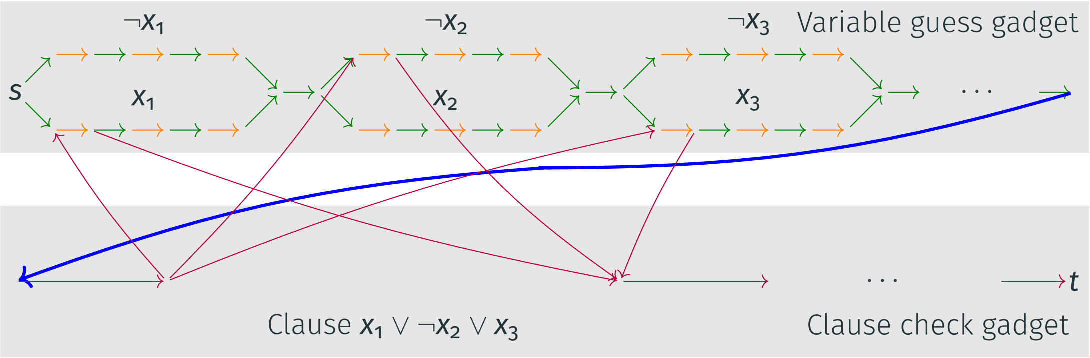

Minimizing #distinct edges forces that all a-edges in the clause checks are revisits

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See [A., Gatterbauer, Makhija, Monet, 2025]

Other Directions and Future Work

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Thanks for your attention!

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