A Circuit-Based Approach to Efficient Enumeration

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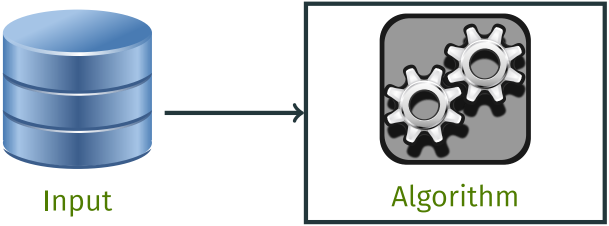
[4]CNRS CRIL

Problem statement

Problem: Enumerating large result sets


Input

Problem: Enumerating large result sets



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Algorithm

A	B	C
a	b	c
a’	b	c
a	b’	c
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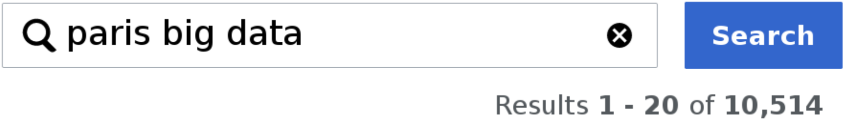

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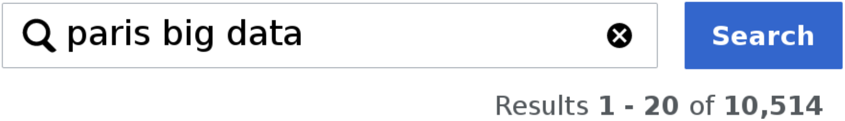

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paris big data

Search

Results **1 - 20** of **10,514**

...

View (previous 20 | next 20) (20 | 50 | 100 | 250 | 500)

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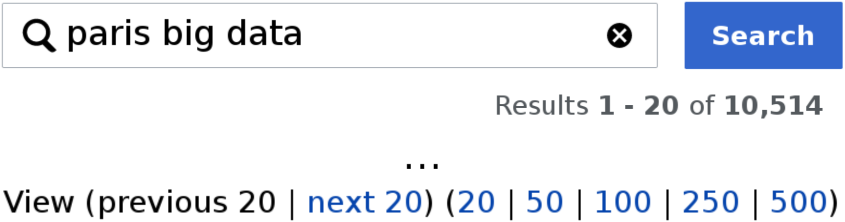

→ Solution: Enumerate solutions one after the other

Enumeration algorithm

Input

Enumeration algorithm

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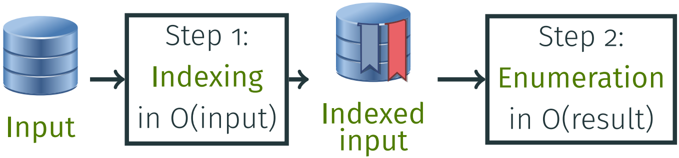

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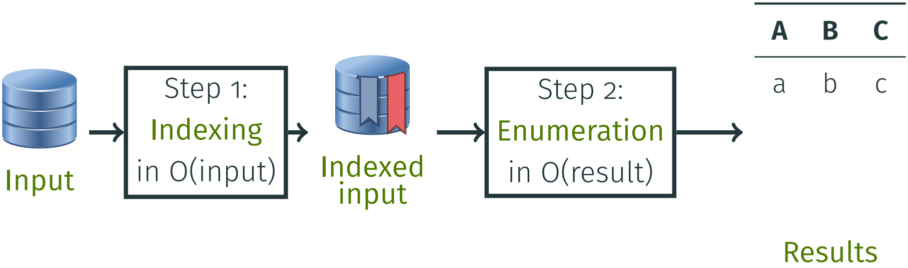

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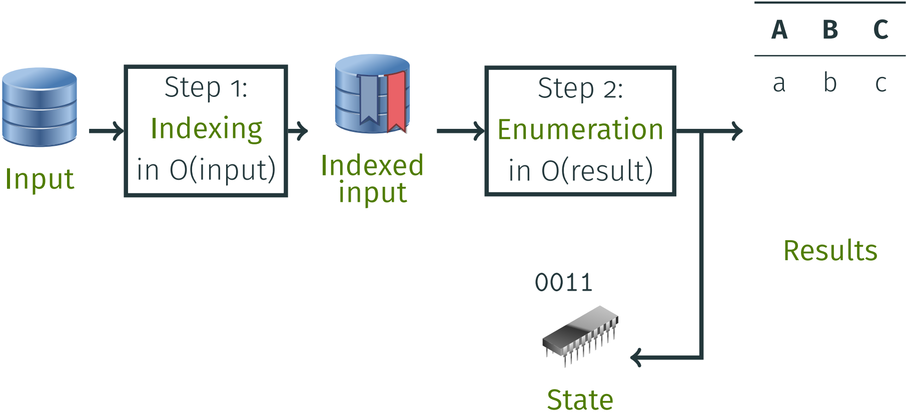

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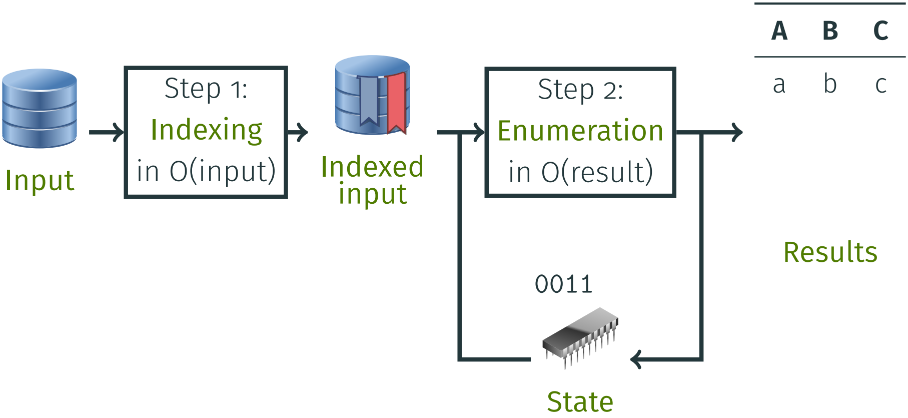

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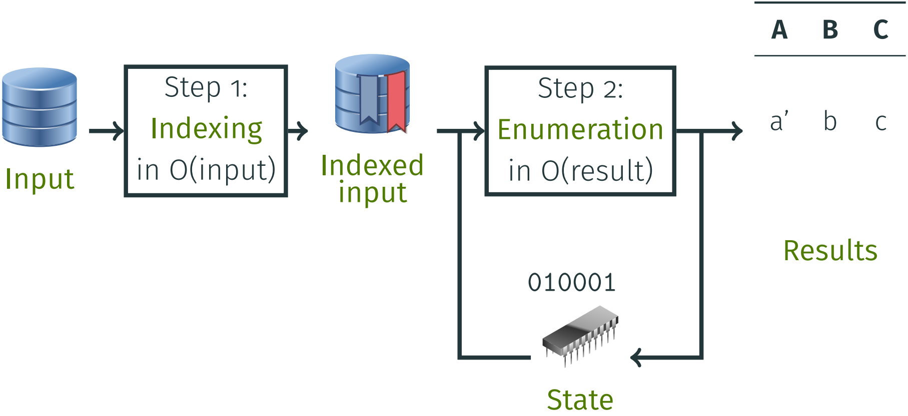

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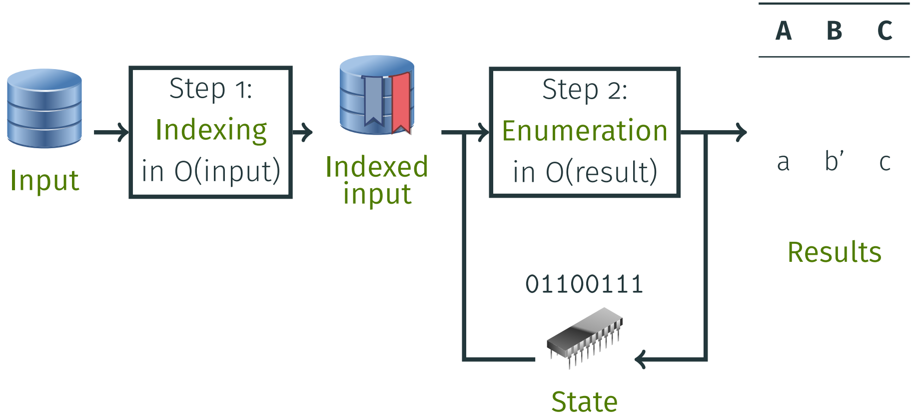

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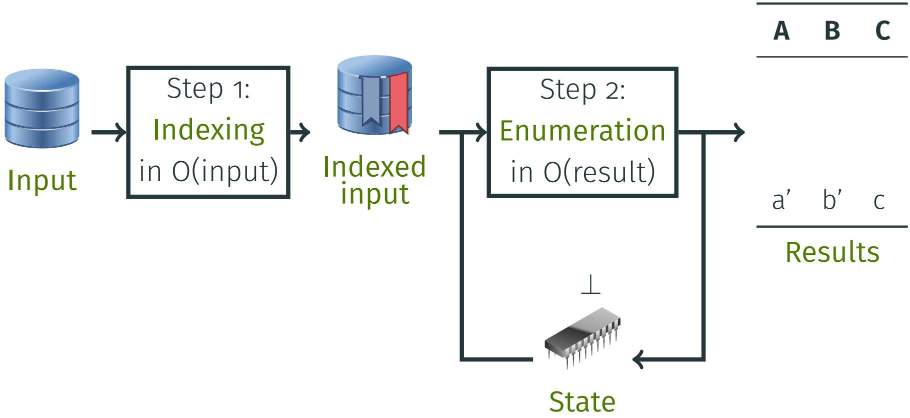
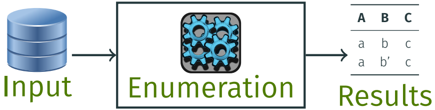


General idea for enumeration

Currently:



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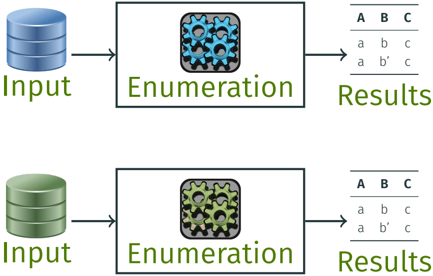

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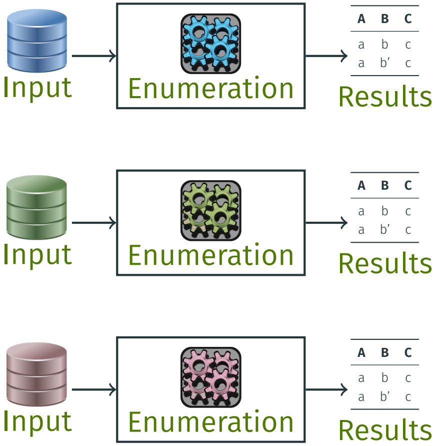

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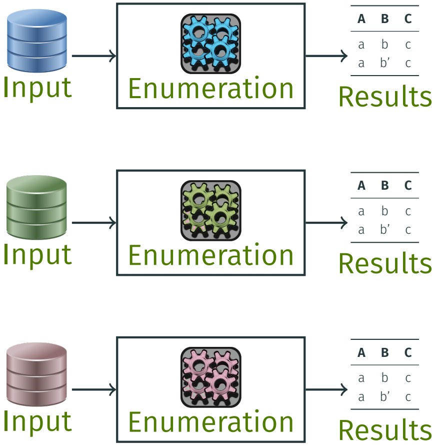

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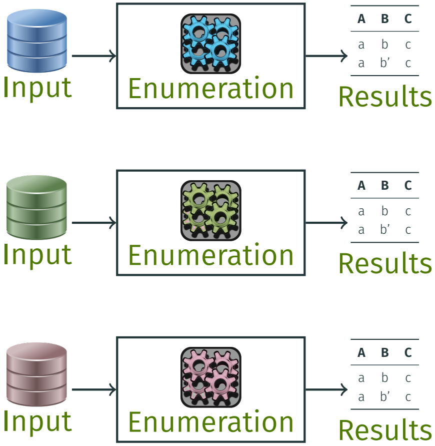

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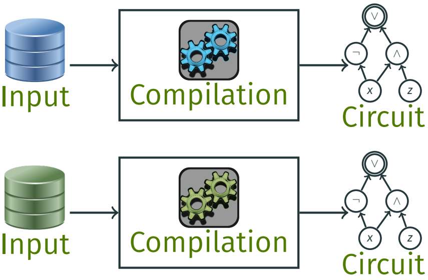

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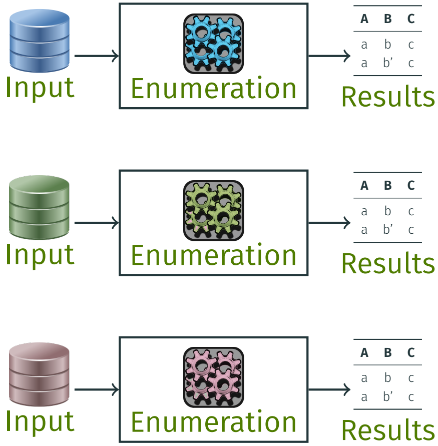

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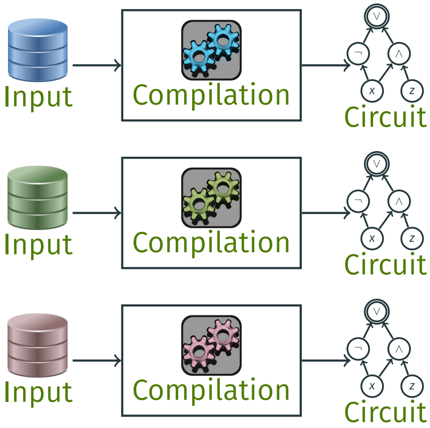

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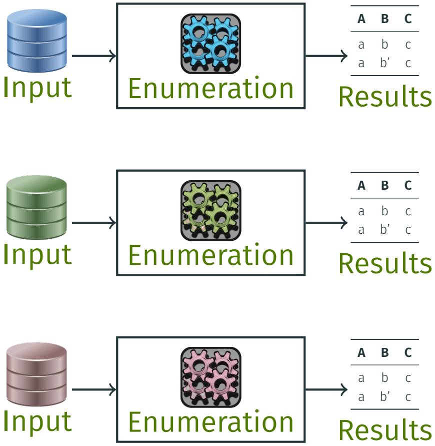

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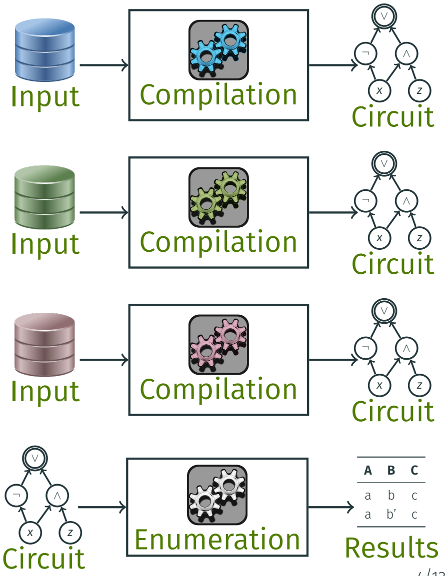
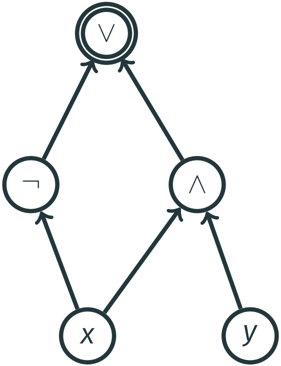
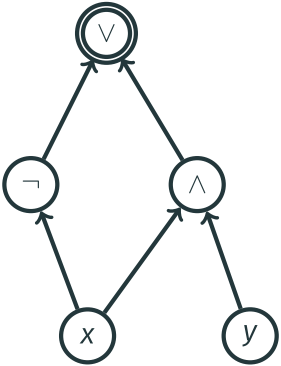


Boolean circuits



- Directed acyclic graph of gates

Boolean circuits



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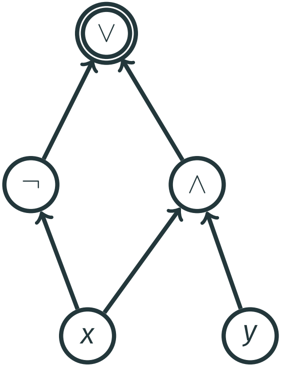
- **Output** gate: 



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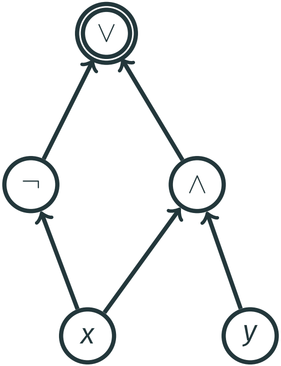

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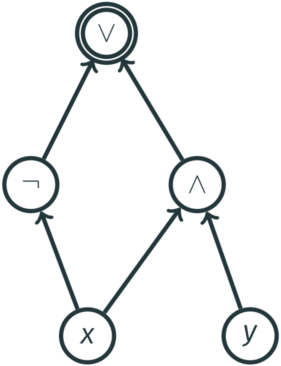

- Directed acyclic graph of **gates**
- **Output** gate: 
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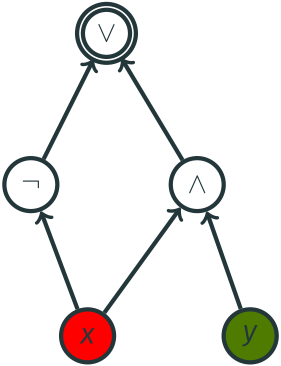
- **Internal** gates: $\lor$ $\land$ $\neg$
- **Valuation**: function from variables to $\{0, 1\}$

  Example: $\nu = \{x \mapsto 0,\ y \mapsto 1\}$...

Boolean circuits



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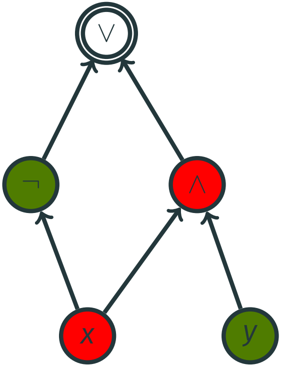

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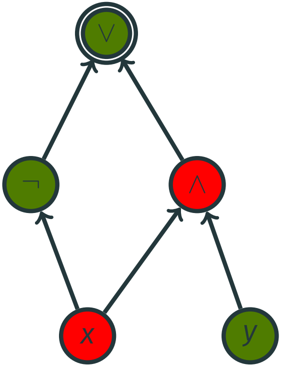

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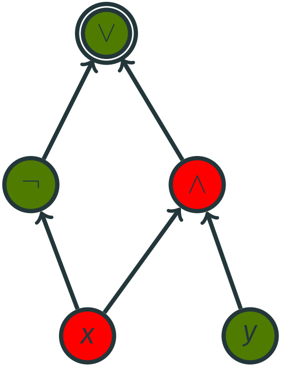

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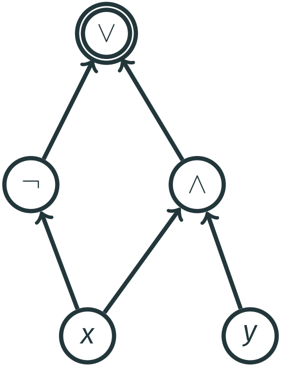

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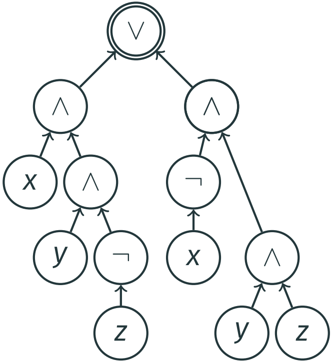
Our task: Enumerate all **satisfying assignments** of an input circuit

Circuit restrictions

d-DNNF:

- $\lor$ are all deterministic:

The inputs are mutually exclusive (= no valuation ν makes two inputs simultaneously evaluate to 1)



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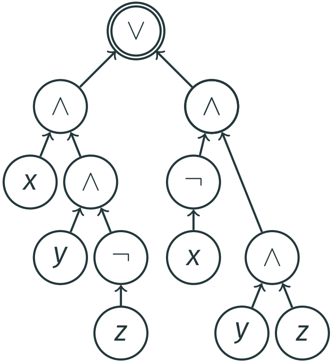

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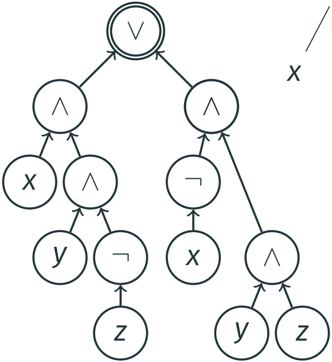
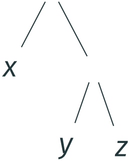
v-tree: ∧-gates follow a tree on the variables





Main results

Theorem

Given a d-DNNF circuit C with a v-tree T, we can enumerate its satisfying assignments with preprocessing linear in $|C| + |T|$ and delay linear in each assignment

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Also: restrict to assignments of constant size $k \in \mathbb{N}$ (at most k variables are set to 1):

Theorem

Given a d-DNNF circuit C with a v-tree T, we can enumerate its satisfying assignments of size $\leq k$ with preprocessing linear in $|C| + |T|$ and constant delay

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- Factorized databases: implicit representation of database tables

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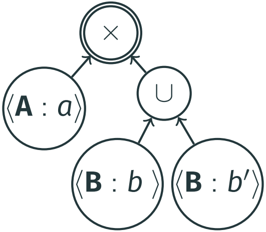
represented by



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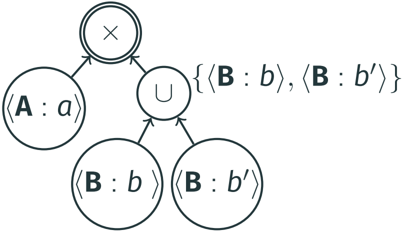

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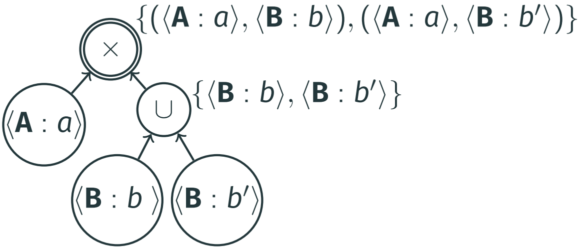

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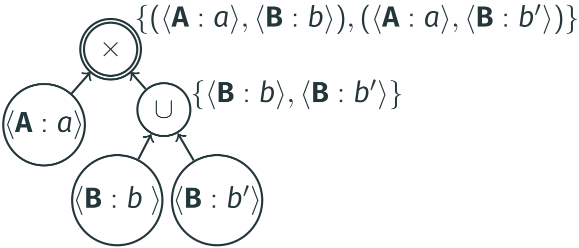

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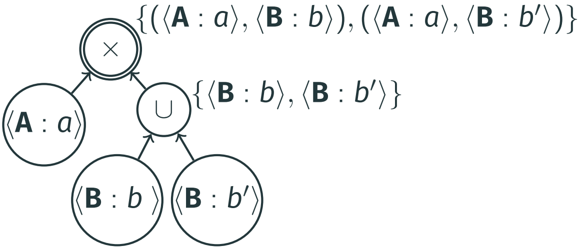

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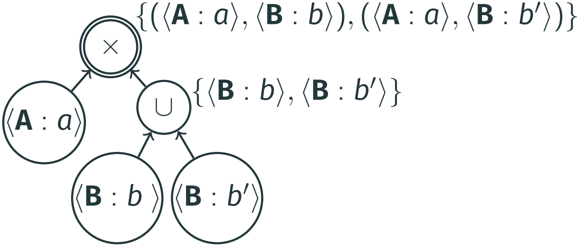

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Theorem (Strenghtens result of [Olteanu and Závodnỳ, 2015])
Given a deterministic factorized representation, we can enumerate its tuples with linear preprocessing and constant delay

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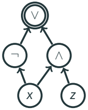
Theorem (Recaptures [Bagan, 2006], [Kazana and Segoufin, 2013])
For any constant $k \in \mathbb{N}$ and fixed MSO query Q, given a database D of treewidth $\leq k$, the results of Q on D can be enumerated with linear preprocessing in D and linear delay in each answer ($\rightarrow$ constant delay for free first-order variables)

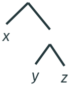
Proof techniques

Proof overview

Preprocessing phase:



Circuit



v-tree

Proof overview

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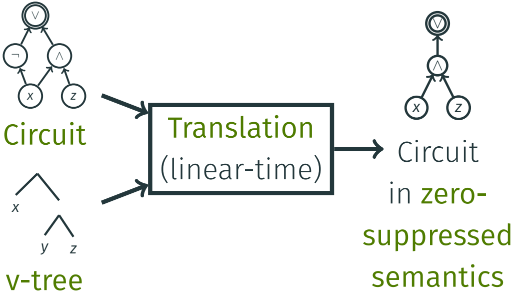

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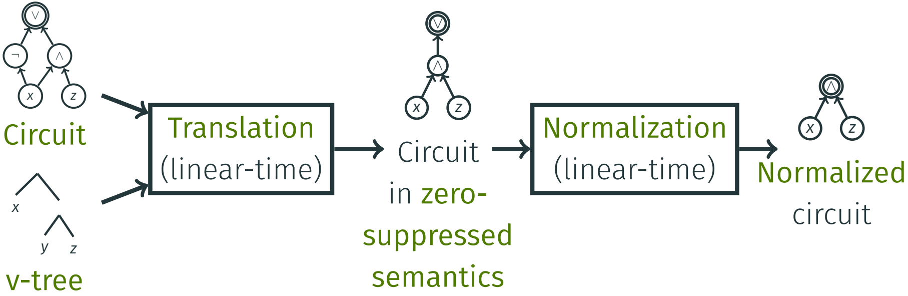

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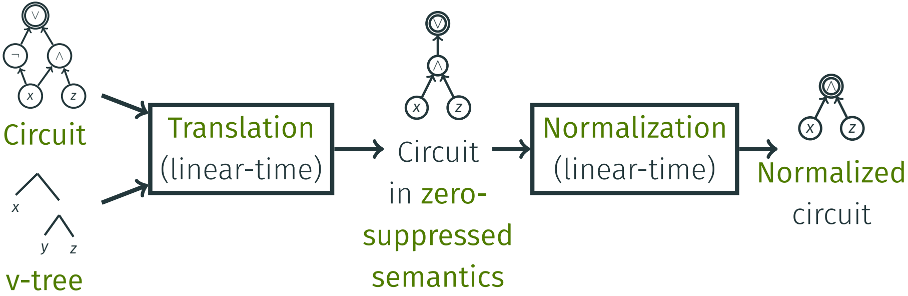

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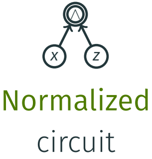

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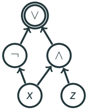

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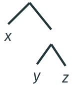
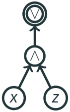


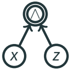
v-tree

Translation
(linear-time)



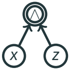
Circuit in zero-suppressed semantics

Normalization
(linear-time)



Normalized circuit

Enumeration phase:



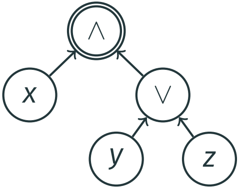
Normalized circuit

Enumeration
(linear delay in each result)

A	B	C
a	b	c
a	b'	c

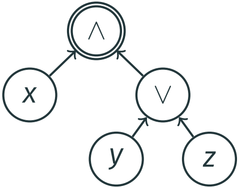
Results

Zero-suppressed semantics



Special zero-suppressed semantics for circuits:

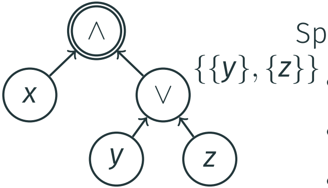
Zero-suppressed semantics



Special zero-suppressed semantics for circuits:

- No NOT-gate
- Each gate captures a set of assignments
- Bottom-up definition with $\times$ and $\cup$

Zero-suppressed semantics



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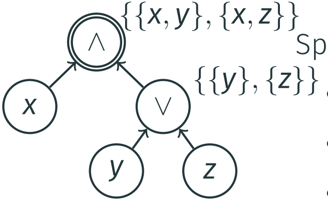

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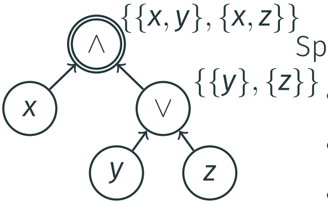

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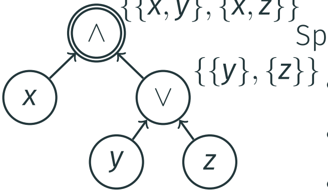

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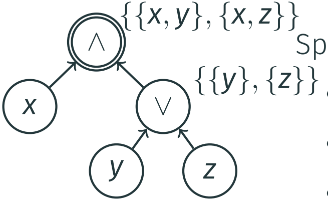
Many equivalent ways to understand this:

- Generalization of factorized representations
- Analogue of zero-suppressed OBDDs (implicit negation)
- Arithmetic circuits: $\times$ and $+$ on polynomials

Zero-suppressed semantics



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Simplification: rewrite circuits to arity-two (fan-in ≤ 2)

Enumerating assignments in the zero-suppressed semantics

Task: Enumerate the elements of the set $S(g)$ captured by a gate g

$\rightarrow$ E.g., for $S(g) = \{\{x, y\}, \{x, z\}\}$, enumerate $\{x, y\}$ and then $\{x, z\}$

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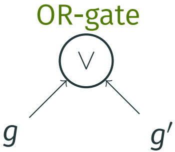

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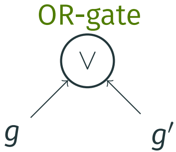

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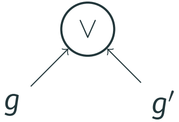
OR-gate



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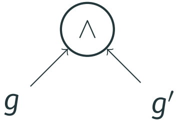
AND-gate



Lexicographic product: enumerate $S(g)$ and for each result t enumerate $S(g')$ and concatenate t with each result

Enumerating assignments in the zero-suppressed semantics

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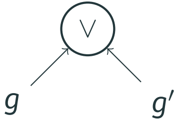
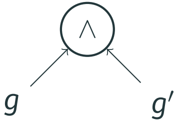


AND-gate



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Decomposability: no duplicates

Conclusion

Summary and conclusion

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- Usual approach: develop enumeration algorithms by hand

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- Theory: handle updates on the structure
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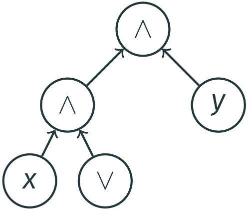
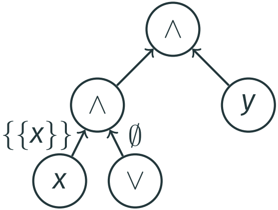
Thanks for your attention!

Normalization: handling $\emptyset$



Normalization: handling ∅



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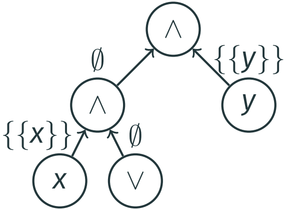

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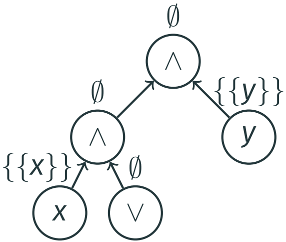

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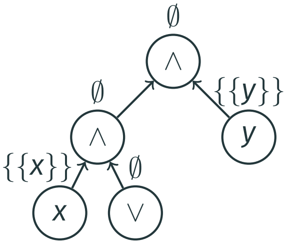

- Problem: if $S(g) = \emptyset$ we waste time

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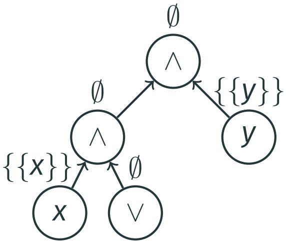
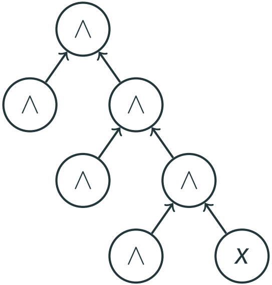


- Problem: if $S(g) = \emptyset$ we waste time
- Solution: compute bottom-up if $S(g) = \emptyset$

Normalization: handling empty assignments



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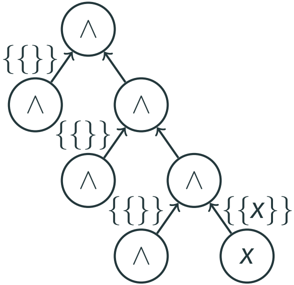

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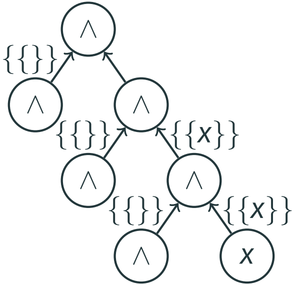

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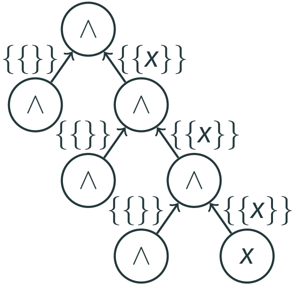

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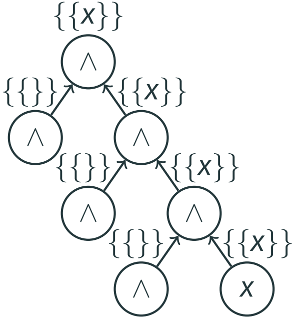

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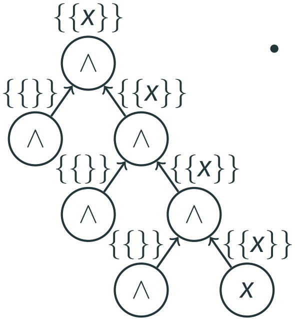

- Problem: if $S(g)$ contains $\{\}$ we waste time in chains of AND-gates

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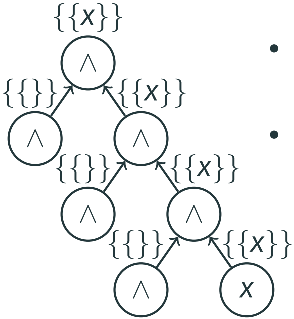

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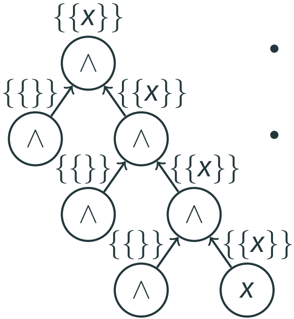

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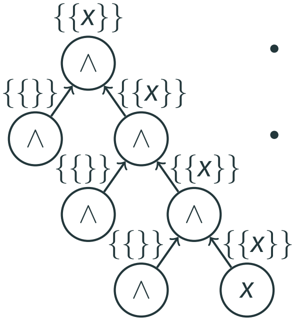

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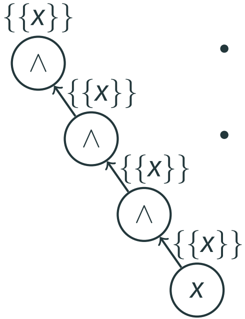

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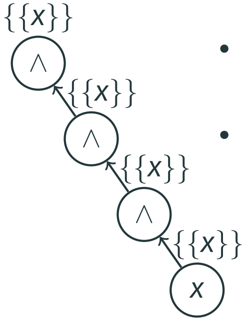

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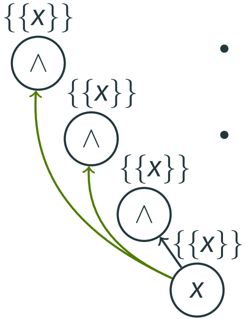

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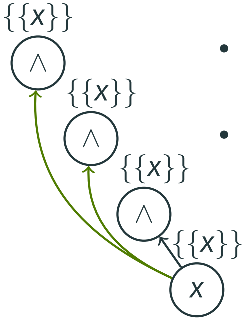

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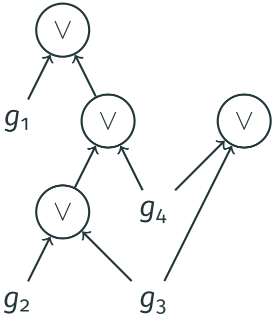
$\rightarrow$ Now, traversing an AND-gate ensures that we make progress: it splits the assignments non-trivially

Normalization: handling OR-hierarchies



- Problem: we waste time in OR-hierarchies to find a reachable exit (non-OR gate)

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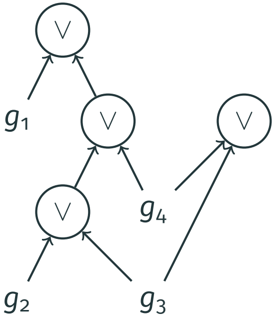

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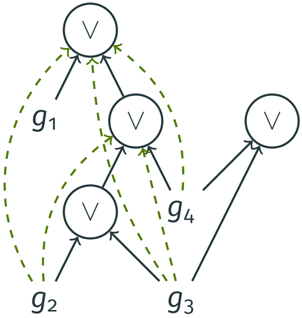

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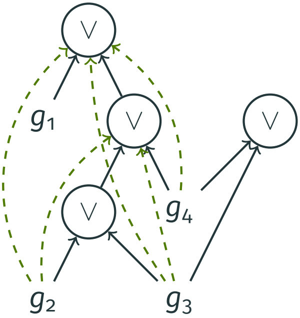

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- **Problem:** must be done in **linear time**

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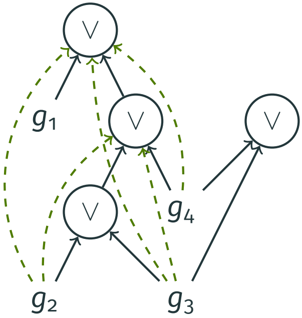

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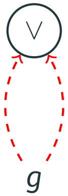
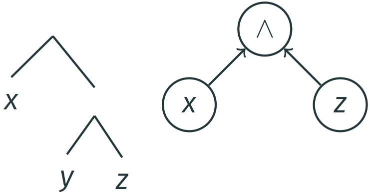
Solution:



- **Determinism** ensures we have a **multitree** (we cannot have the pattern at the right)
- **Custom** constant-delay reachability index for multitrees

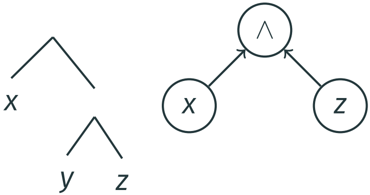
Translating to zero-suppressed semantics

- This is where we use the v-tree



Translating to zero-suppressed semantics

- This is where we use the v-tree
- Add explicitly untested variables



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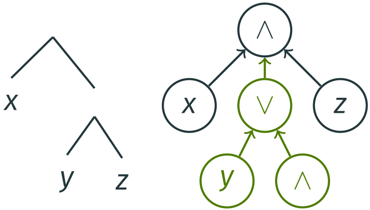

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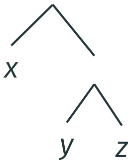

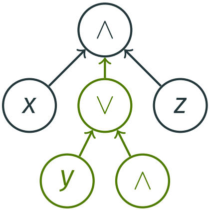

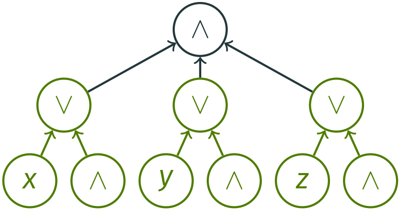

- **Problem:** quadratic blowup

Translating to zero-suppressed semantics

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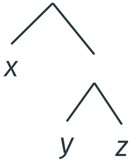

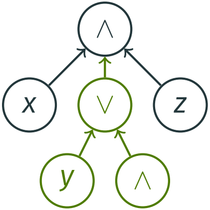

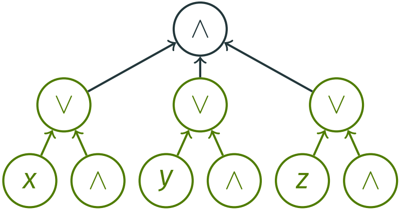

- **Problem:** quadratic blowup
- **Solution:**
  - **Order** $<$ on variables in the v-tree ($x < y < z$)
  - **Interval** $[x, z]$
  - **Range gates** to denote $\bigvee[x, z]$ in constant space

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