Finding Topological Sorts in a Regular Language

Antoine Amarilli

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Télécom ParisTech

The Constrained Topological Sorting Problem
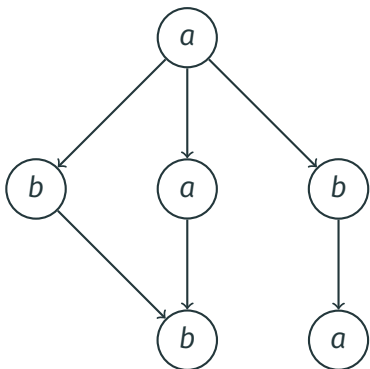
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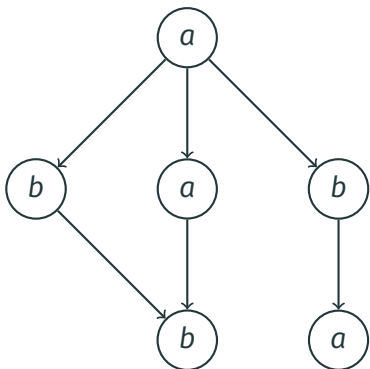## The Constrained Topological Sorting Problem

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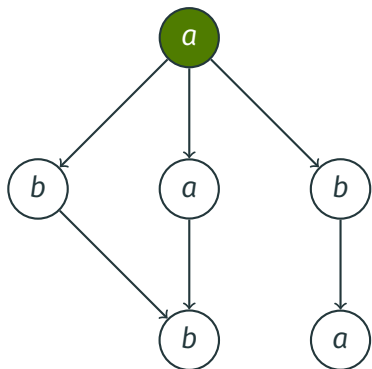
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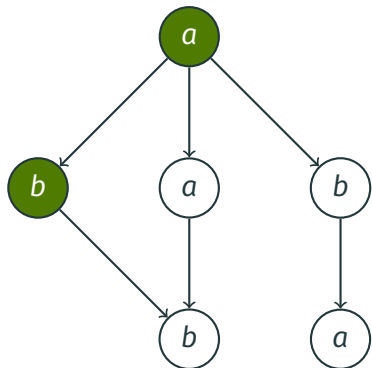 - **Input:** DAG whose vertices carry a label from Σ

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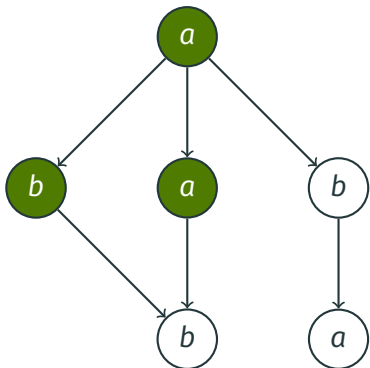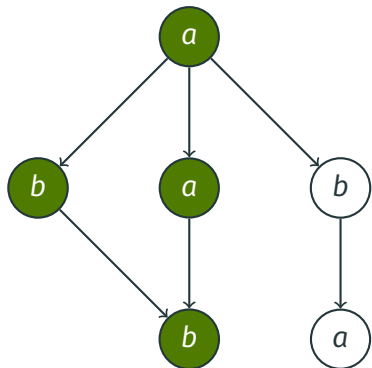
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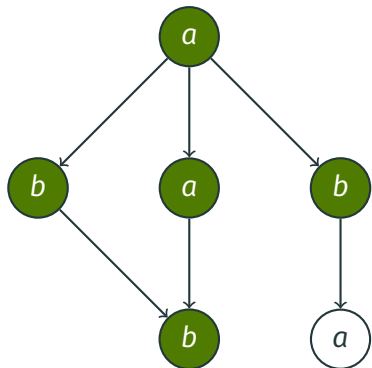
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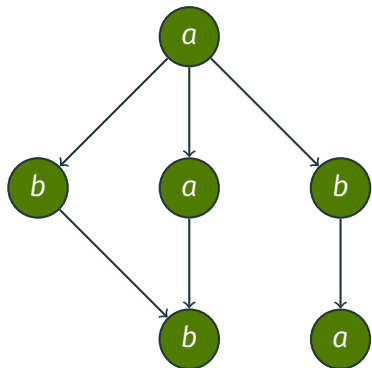 - **Output:** is there a **topological sort** achieving a word of L?

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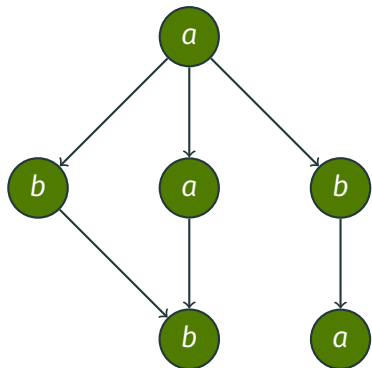 - **Output:** is there a **topological sort** achieving a word of L?

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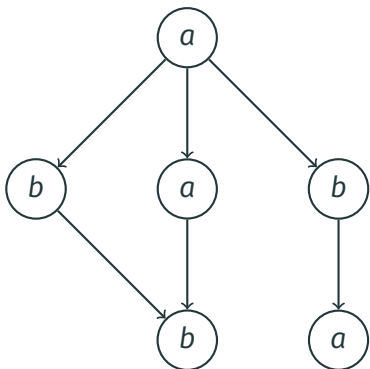 - **Output:** is there a **topological sort** achieving a word of *L*?

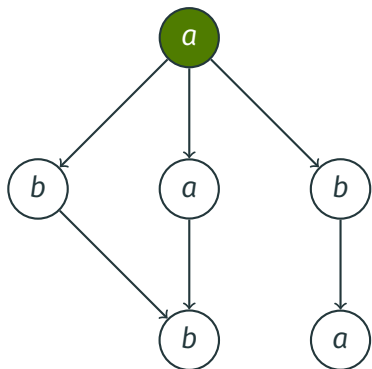# The Constrained Topological Sorting Problem

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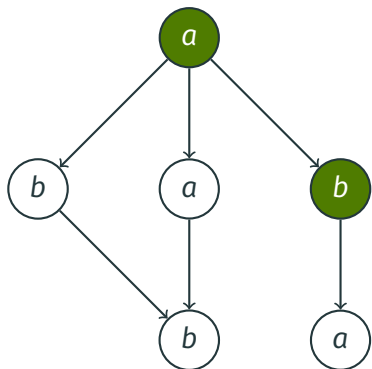 - **Output:** is there a **topological sort** achieving a word of L?

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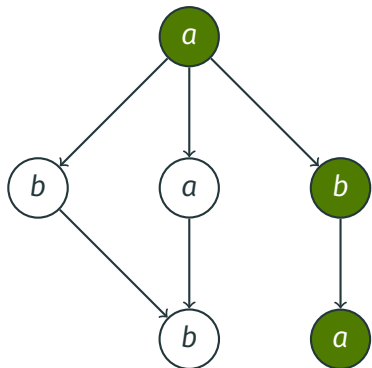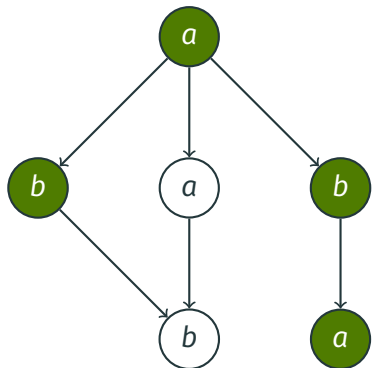
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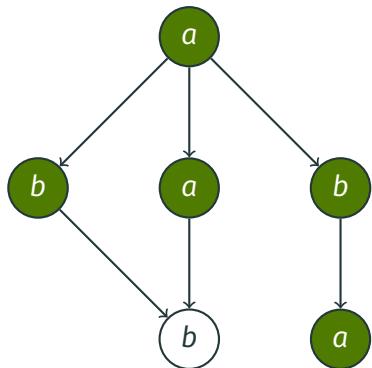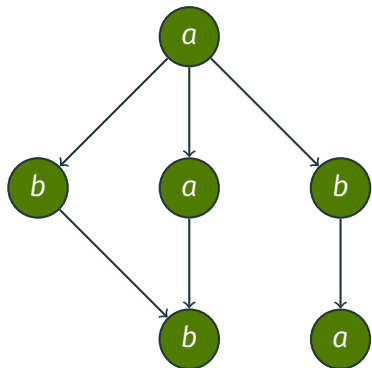
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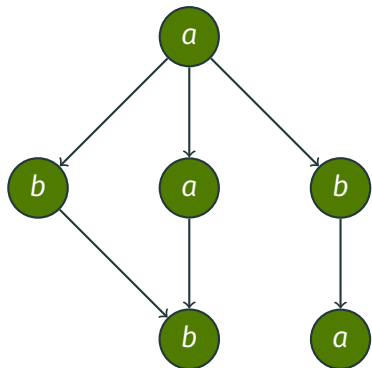 - **Output:** is there a topological sort achieving a word of L?

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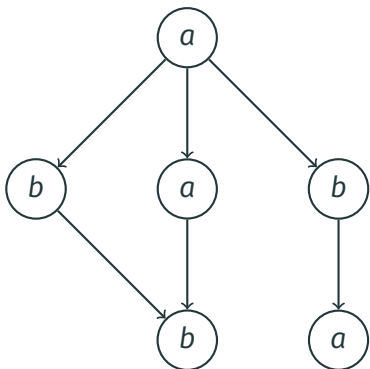 - **Output:** is there a **topological sort** achieving a word of L?

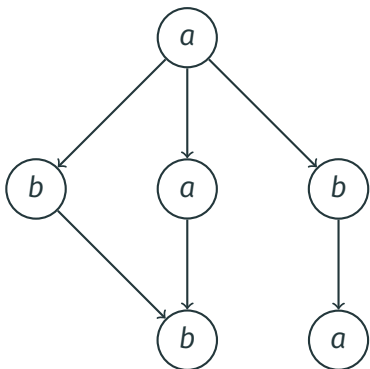# The Constrained Topological Sorting Problem

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- **Question:** for which languages is $\mathrm{CTS}(L)$ tractable?
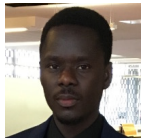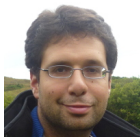
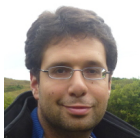- Originally with Daniel Deutch, Lamine Ba, and Pierre Senellart

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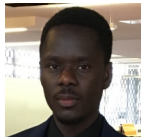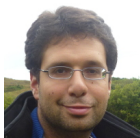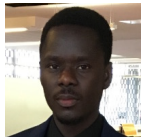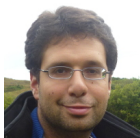- Now working on this with Charles Paperman

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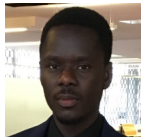- Now working on this with Charles Paperman
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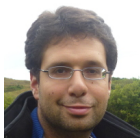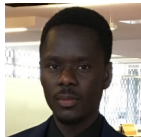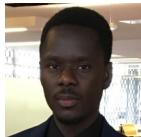- Current results:

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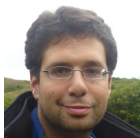$(a|ab)^*$ is

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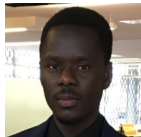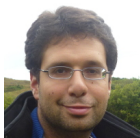- Now working on this with Charles Paperman
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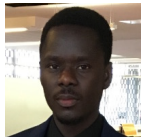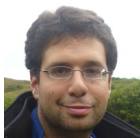- Now working on this with Charles Paperman
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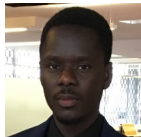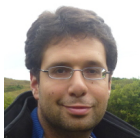- Now working on this with Charles Paperman
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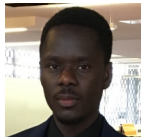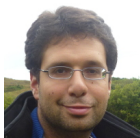- Current results:

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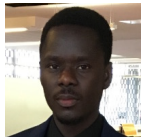- Current results:

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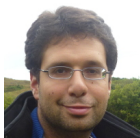- Current results:

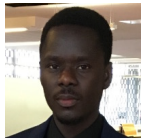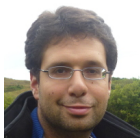$(ab)^*$ is NP-hard $(aaabab)^*$ is NP-hard

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$(a\|ab)^*$ is NP-hard	ba^*ba^*b is in NL	
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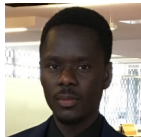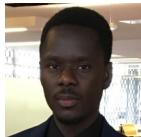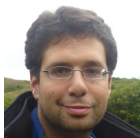$(ab)^*$ is NP-hard $(aaabab)^*$ is NP-hard   Groups are PTIME?

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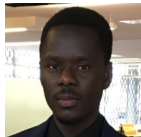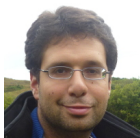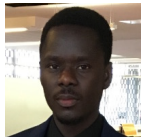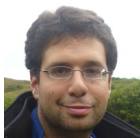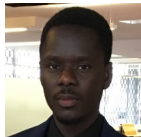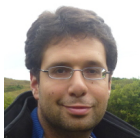
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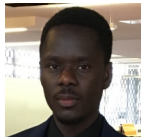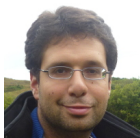$(a|ab)^*$ is NP-hard ba^*ba^*b is in NL $(a|bb)^*$ is ?!?

$(aa|bb)^*$ is NP-hard $(ab)^* | \Sigma^* aa \Sigma^*$ is in NL   Dyck language?

- Originally with Daniel Deutch, Lamine Ba, and Pierre Senellart
- Now working on this with Charles Paperman
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$(ab)^*$ is NP-hard $\qquad$ $(aaabab)^*$ is NP-hard $\qquad$ Groups are PTIME?

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$(aa|bb)^*$ is NP-hard $\qquad$ $(ab)^* \mid \Sigma^*aa\Sigma^*$ is in NL $\qquad$ Dyck language?

Can we show a **dichotomy**: $\mathrm{CTS}(L)$ is either PTIME or NP-hard?

Our Current Results

- Originally with Daniel Deutch, Lamine Ba, and Pierre Senellart
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- Current results:

$(ab)^*$ is NP-hard	$(aaabab)^*$ is NP-hard	Groups are PTIME?
$(a\|ab)^*$ is NP-hard	ba^*ba^*b is in NL	$(a\|bb)^*$ is ?!?
$(aa\|bb)^*$ is NP-hard	$(ab)^* \mid \Sigma^*aa\Sigma^*$ is in NL	Dyck language?

Can we show a **dichotomy**: $\text{CTS}(L)$ is either PTIME or NP-hard?

More info in our **preprint**: `https://arxiv.org/abs/1707.04310`