Finding Topological Sorts in a Regular Language

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Diagram:

- Vertices labeled with $a$, $b$, $a$, $b$, $b$, $a$, $a$.
- Edges from $b$ to $a$, $b$, $a$, $b$.
- Edges from $a$ to $b$, $a$.
- Edges from $b$ to $b$, $a$.
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- Consider the problem $CTS(L)$:
  - **Input:** DAG whose vertices carry a label from $\Sigma$
  - **Output:** is there a topological sort achieving a word of $L$?
- **Question:** for which languages is $CTS(L)$ tractable?
Our Current Results

• Originally with Daniel Deutch, Lamine Ba, and Pierre Senellart

Current results:

- $(ab)^* \text{ is NP-hard}$
- $(a|bb)^* \text{ is NP-hard}$
- $(a|bb)^* \text{ is NP-hard}$
- $(ab)^* | \Sigma^* \text{ is in NL}$
- $(a|bb)^* \Sigma^* \text{ is ?!?}$

Groups are PTIME?

Dyck language?

Can we show a dichotomy: CTS$(L)$ is either PTIME or NP-hard?

More info in our preprint: https://arxiv.org/abs/1707.04310
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\(ba^*ba^*b\) is in NL

\((a|bb|^*)\) is NP-hard

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