Out-of-Order Membership in Regular Languages

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Original presentation by Sebastien Felippe Labbe, ICALP'26

The Membership Problem

Definition

Fix a language $L \subseteq \Sigma^*$. Given a word w, decide whether $w \in L$.

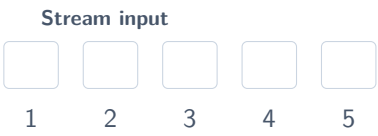
Example

$$\Sigma = \{a, b\} \qquad L = \Sigma^* aa \Sigma^*$$

$$aabab \in L \qquad ababa \notin L$$

Membership: Left-to-Right Streaming

Fixed: $\Sigma = \{a, b\}$ $\quad L = \Sigma^* aa \Sigma^*$



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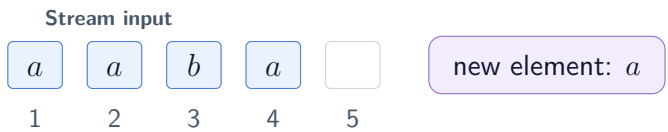
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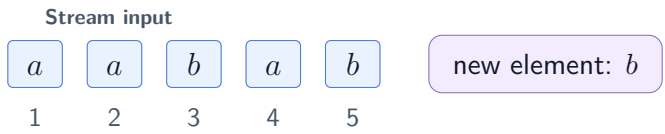
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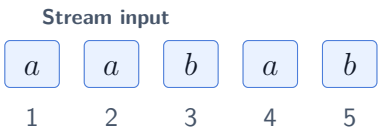
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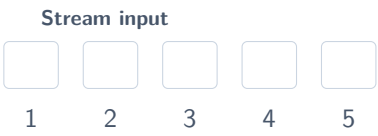


Decision: Is the word $aabab$ in L?

Membership: Out-of-Order Streaming

Fixed: $\Sigma = \{a, b\}$ $\quad L = \Sigma^* aa \Sigma^*$

Setting: adversarial arrival order



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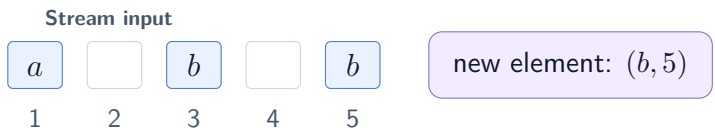
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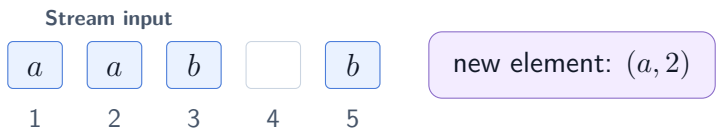
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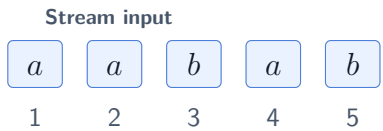
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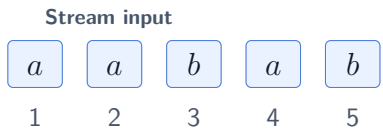


new element: $(a, 4)$

Membership: Out-of-Order Streaming

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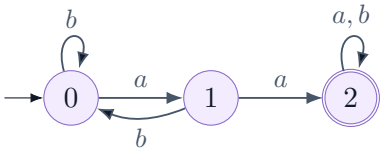
Setting: adversarial arrival order



Decision: Is the word $aabab$ in L?

Regular Languages

We focus on regular languages, which are recognized by deterministic finite automata.



$$L = \Sigma^* aa \Sigma^*$$

Each letter defines a transition function.

	a	b
0	1	0
1	2	0
2	2	2

letter a

$0 \mapsto 1$
$1 \mapsto 2$
$2 \mapsto 2$

letter b

$0 \mapsto 0$
$1 \mapsto 0$
$2 \mapsto 2$

Out-of-Order Membership: Regular Languages, Data complexity

- **Model:** out-of-order membership.
- **Languages:** regular languages.
- **Complexity:** data complexity.
  - **Fixed:** the language L is fixed.
  - **Input:** a word w of length n, with n known in advance.
  - We study **time per element** and **space** as functions of n.

General Upper Bound: Linear Space

Fixed: $\Sigma = \{a, b\} \quad L = \Sigma^* aa \Sigma^*$

Setting: adversarial arrival order

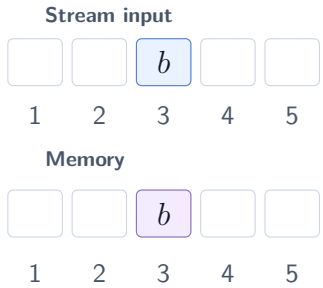


Theorem: Out-of-order membership in any fixed regular language is in $O(n)$ time per element and $O(n)$ space.

General Upper Bound: Linear Space

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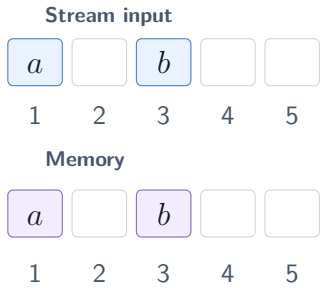
new element: $(b, 3)$

Theorem: Out-of-order membership in any fixed regular language is in $O(n)$ time per element and $O(n)$ space.

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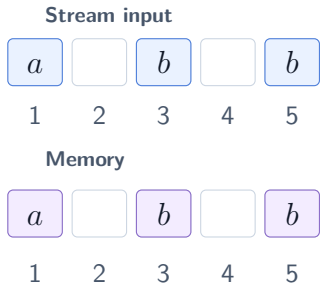
new element: $(a, 1)$

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General Upper Bound: Linear Space

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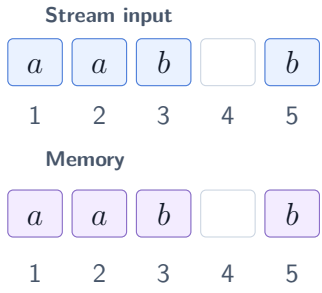
new element: $(b, 5)$

Theorem: Out-of-order membership in any fixed regular language is in $O(n)$ time per element and $O(n)$ space.

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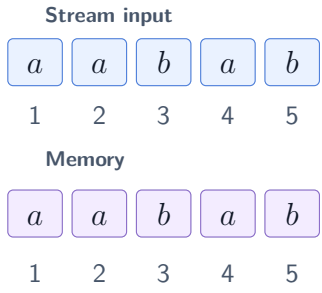
new element: $(a, 2)$

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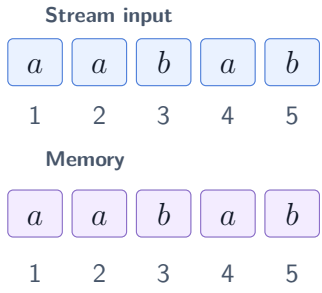
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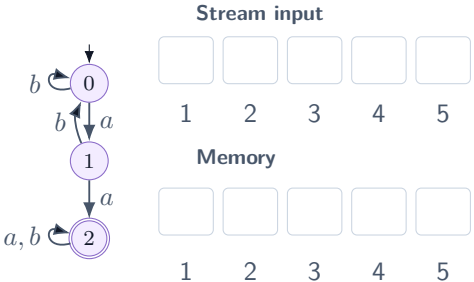
Decision: Run the automaton on the memorized word $aabab$.

Theorem: Out-of-order membership in any fixed regular language is in $O(n)$ time per element and $O(n)$ space.

General Upper Bound: Constant Time per Element

Fixed: $\Sigma = \{a, b\}$ $\quad L = \Sigma^* aa \Sigma^*$

Setting: adversarial arrival order

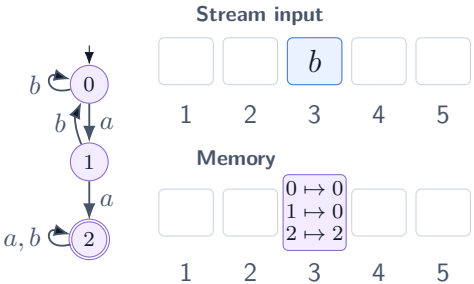


Store transition functions.

Theorem: Out-of-order membership in any fixed regular language is in $O(1)$ time per element and $O(n)$ space.

General Upper Bound: Constant Time per Element

Fixed: $\Sigma = \{a, b\}$ $\quad L = \Sigma^* aa \Sigma^*$ **Setting:** adversarial arrival order



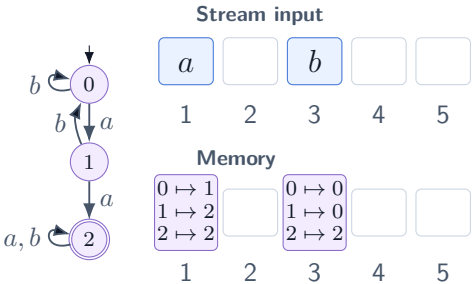
new element: $(b, 3)$

Store transition functions.

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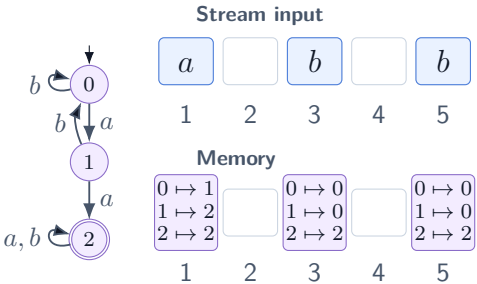
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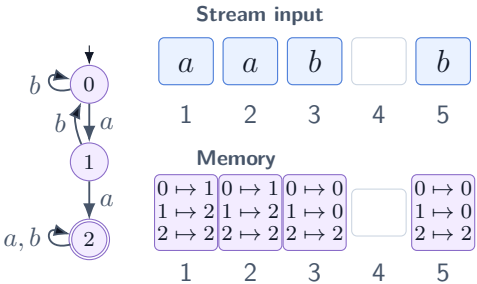
new element: $(b, 5)$

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General Upper Bound: Constant Time per Element

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new element: $(a, 2)$

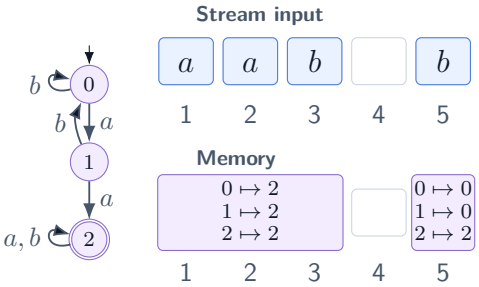
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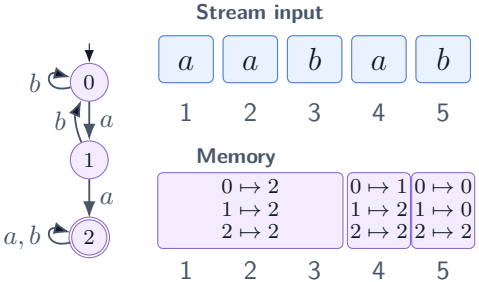
Compose adjacent functions.

Theorem: Out-of-order membership in any fixed regular language is in $O(1)$ time per element and $O(n)$ space.

General Upper Bound: Constant Time per Element

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new element: $(a, 4)$

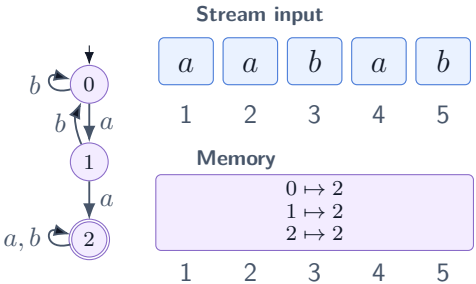
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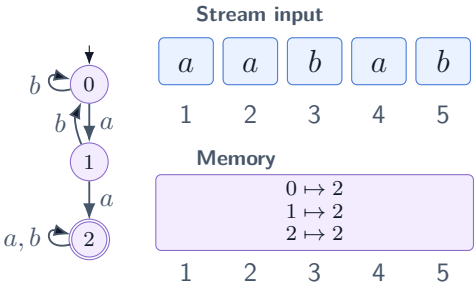
Final block summarizes word.

Theorem: Out-of-order membership in any fixed regular language is in $O(1)$ time per element and $O(n)$ space.

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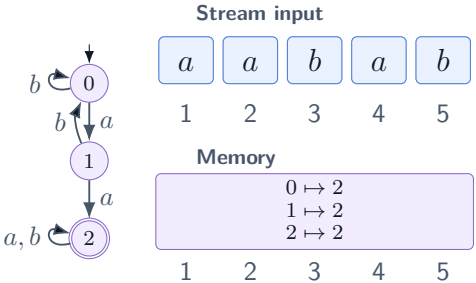
Decision: Accept iff initial state is mapped to final state

Theorem: Out-of-order membership in any fixed regular language is in $O(1)$ time per element and $O(n)$ space.

General Upper Bound: Constant Time per Element

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Setting: adversarial arrival order



Final block summarizes word.

Decision: Accept iff initial state is mapped to final state

Theorem: Out-of-order membership in any fixed regular language is in $O(1)$ time per element and $O(n)$ **bits of memory**.

Space Regimes: Constant vs. Linear

General upper bound: $O(1)$ time per element and $O(n)$ bits of memory.

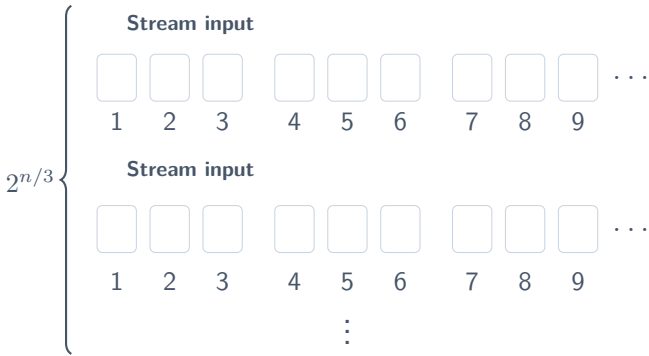
Can this space bound be improved?

- for every language?
- or at least for some languages?

Example: $\Sigma^* aa \Sigma^*$ needs $\Omega(n)$ space

Fixed: $\Sigma = \{a, b\}$ $\quad L = \Sigma^* aa \Sigma^*$

Setting: adversarial arrival order



Theorem: Out-of-order membership in $\Sigma^* aa \Sigma^*$ requires $\Omega(n)$ space.

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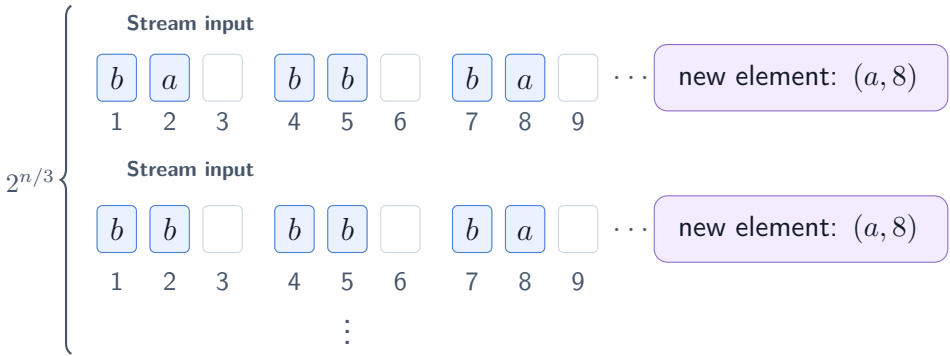
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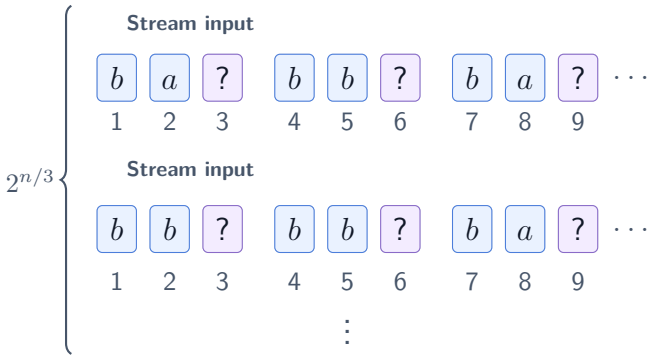
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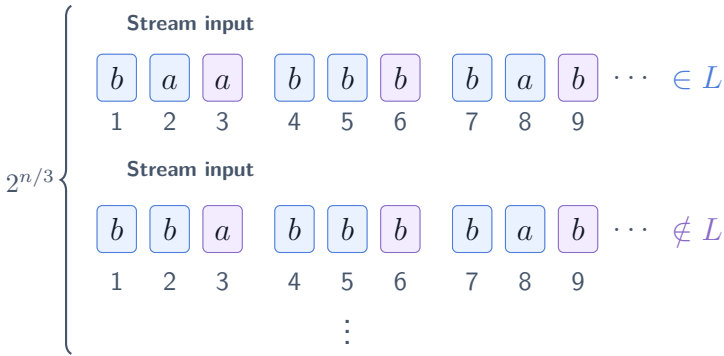
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Theorem: Out-of-order membership in $\Sigma^* aa \Sigma^*$ requires $\Omega(n)$ space.

Space Regimes: Constant vs. Linear

General upper bound: $O(1)$ time per element and $O(n)$ bits of memory.

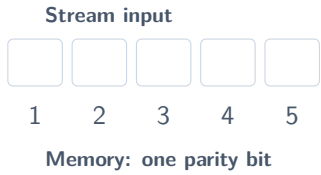
Can this space bound be improved?

- for every language?
  - No: For consecutive a's $\left(\Sigma^* aa \Sigma^*\right)$, there is an $\Omega(n)$ lower bound.
- or at least for some languages?

Example: $b^*(ab^*ab^*)^*$ is in $O(1)$ space

Fixed: $\Sigma = \{a, b\}$ $\quad L = b^*(ab^*ab^*)^*$

Setting: adversarial arrival order

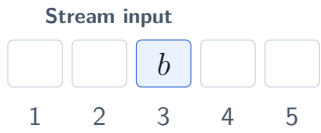


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Example: $b^*(ab^*ab^*)^*$ is in $O(1)$ space

Fixed: $\Sigma = \{a, b\}$ $\quad L = b^*(ab^*ab^*)^*$

Setting: adversarial arrival order



new element: $(b, 3)$

On a: flip the bit.

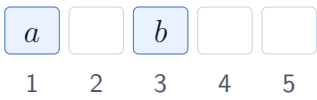
Theorem: Out-of-order membership in $b^*(ab^*ab^*)^*$ is in $O(1)$ space.

Example: $b^*(ab^*ab^*)^*$ is in $O(1)$ space

Fixed: $\Sigma = \{a, b\}$ $\quad L = b^*(ab^*ab^*)^*$

Setting: adversarial arrival order

Stream input



new element: $(a, 1)$

Memory: one parity bit

odd

On a: flip the bit.

Theorem: Out-of-order membership in $b^*(ab^*ab^*)^*$ is in $O(1)$ space.

Example: $b^*(ab^*ab^*)^*$ is in $O(1)$ space

Fixed: $\Sigma = \{a, b\}$ $\quad L = b^*(ab^*ab^*)^*$ **Setting:** adversarial arrival order

Stream input



new element: $(b, 5)$

Memory: one parity bit

odd

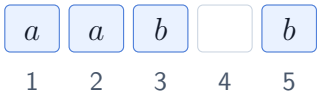
On a: flip the bit.

Theorem: Out-of-order membership in $b^*(ab^*ab^*)^*$ is in $O(1)$ space.

Example: $b^*(ab^*ab^*)^*$ is in $O(1)$ space

Fixed: $\Sigma = \{a, b\}$ $\quad L = b^*(ab^*ab^*)^*$ **Setting:** adversarial arrival order

Stream input



new element: $(a, 2)$

Memory: one parity bit

even

On a: flip the bit.

Theorem: Out-of-order membership in $b^*(ab^*ab^*)^*$ is in $O(1)$ space.

Example: $b^*(ab^*ab^*)^*$ is in $O(1)$ space

Fixed: $\Sigma = \{a, b\}$ $L = b^*(ab^*ab^*)^*$

Setting: adversarial arrival order

Stream input

a	a	b	a	b
1	2	3	4	5

new element: $(a, 4)$

Memory: one parity bit

odd

On a: flip the bit.

Decision: Accept iff memory state is even

Theorem: Out-of-order membership in $b^*(ab^*ab^*)^*$ is in $O(1)$ space.

Space Regimes: Constant vs. Linear

General upper bound: $O(1)$ time per element and $O(n)$ bits of memory.

Can this space bound be improved?

- for every language?
  - No: For consecutive a's $\left(\Sigma^* aa \Sigma^*\right)$, there is an $\Omega(n)$ lower bound.
- or at least for some languages?
  - Yes: For parity $\left(b^*(ab^*ab^*)^*\right)$, there is an $O(1)$ upper bound.

Main Question: Classify Space

Problem

Which **fixed regular languages** have which **space data complexity** for out-of-order membership?

At least two behaviours:

$$L = b^*(ab^*ab^*)^* \quad : \quad O(1) \text{ space}$$
$$L = \Sigma^* aa \Sigma^* \quad : \quad \Omega(n) \text{ space}$$

Which languages are constant? Which are linear? Are there other regimes?

Problem Variants

Out-of-order membership variant	Position content
Original problem	$\Sigma \quad (a, 1)$

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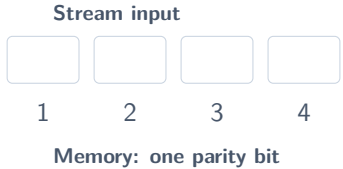
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Monoid Variant: Example Stream

Fixed: $\Sigma = \{a, b\}$ $\quad L = b^*(ab^*ab^*)^*$

Setting: adversarial arrival order, Σ^* position content



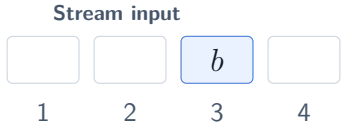
Flip if the new word has an odd number of a's.

Theorem: The monoid variant for $b^*(ab^*ab^*)^*$ is in $O(1)$ space.

Monoid Variant: Example Stream

Fixed: $\Sigma = \{a, b\}$ $\quad L = b^*(ab^*ab^*)^*$

Setting: adversarial arrival order, Σ^* position content



new element: $(b, 3)$

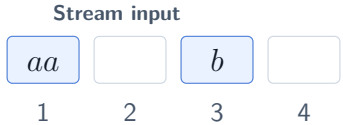
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Monoid Variant: Example Stream

Fixed: $\Sigma = \{a, b\}$ $\quad L = b^*(ab^*ab^*)^*$

Setting: adversarial arrival order, Σ^* position content



new element: $(aa, 1)$

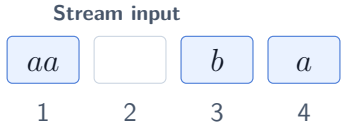
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Theorem: The monoid variant for $b^*(ab^*ab^*)^*$ is in $O(1)$ space.

Monoid Variant: Example Stream

Fixed: $\Sigma = \{a, b\}$ $\quad L = b^*(ab^*ab^*)^*$

Setting: adversarial arrival order, Σ^* position content



new element: $(a, 4)$

Flip if the new word has an odd number of a's.

Theorem: The monoid variant for $b^*(ab^*ab^*)^*$ is in $O(1)$ space.

Monoid Variant: Example Stream

Fixed: $\Sigma = \{a, b\}$ $\quad L = b^*(ab^*ab^*)^*$

Setting: adversarial arrival order, Σ^* position content

Stream input

aa	ε	b	a
1	2	3	4

Memory: one parity bit

odd

new element: $(\varepsilon, 2)$

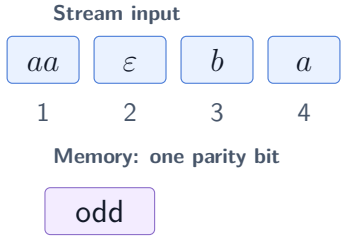
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Monoid Variant: Example Stream

Fixed: $\Sigma = \{a, b\}$ $\quad L = b^*(ab^*ab^*)^*$

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Flip if the new word has an odd number of a's.

Decision: Accept iff final memory state is even

Theorem: The monoid variant for $b^*(ab^*ab^*)^*$ is in $O(1)$ space.

Monoid Variant: $O(1)$ Space

Definition

$L \subseteq \Sigma^*$ is in $\mathbf{Com}$ if letters can always be swapped without changing membership:

$$\forall u, v \in \Sigma^*,\ \forall a, b \in \Sigma, \quad uabv \in L \iff ubav \in L.$$

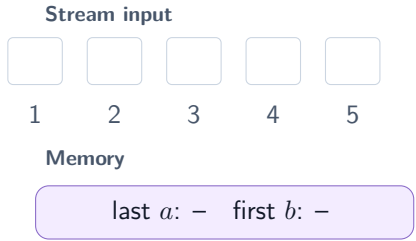
Theorem

A regular language has constant space data complexity for out-of-order membership in the monoid variant if and only if it is in $\mathbf{Com}$.

Monoid Variant: a^*b^* is in $O(\log n)$ Space

Fixed: $\Sigma = \{a, b\}$ $L = a^*b^*$

Setting: adversarial arrival order, Σ^* position content

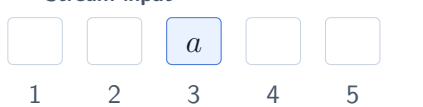


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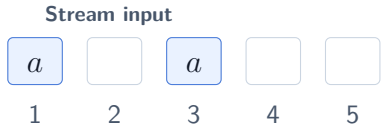


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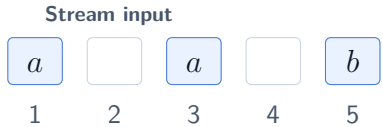
new element: $(a, 1)$

Theorem: The monoid variant for a^*b^* is in $O(\log n)$ space.

Monoid Variant: a^*b^* is in $O(\log n)$ Space

Fixed: $\Sigma = \{a, b\}$ $\quad L = a^*b^*$

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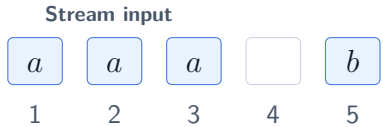


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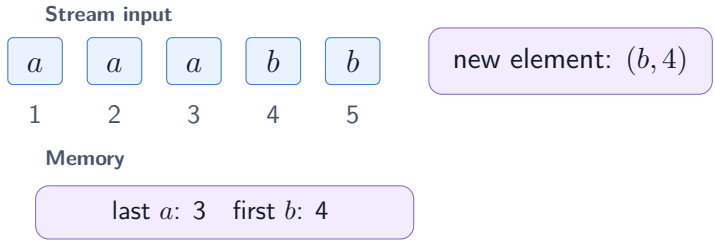


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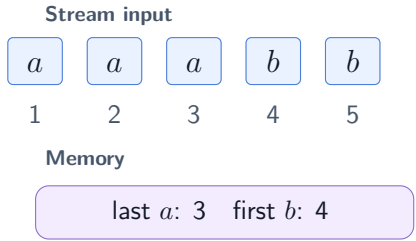


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Monoid Variant: a^*b^* is in $O(\log n)$ Space

Fixed: $\Sigma = \{a, b\}$ $\quad L = a^*b^*$

Setting: adversarial arrival order, Σ^* position content



Decision: Accept iff last $a <$ first b

Theorem: The monoid variant for a^*b^* is in $O(\log n)$ space.

Monoid Variant: $O(\log n)$ Space

Definition

$L \subseteq \Sigma^*$ is in $\mathbf{FL}$ if there exists a constant k such that membership is unchanged when we keep only the first k and last k positions of each letter:

$$w \in L \iff \mathrm{FL}_k(w) \in L.$$

Theorem

A regular language has logarithmic space data complexity for out-of-order membership in the monoid variant if and only if $L \in \mathbf{FL} \vee \mathbf{Com}$.

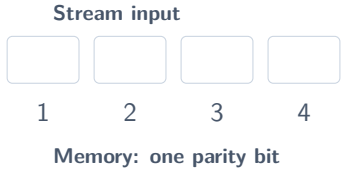
Problem variants

Out-of-order membership variant	position content
Original problem	$\Sigma \quad (a, 1)$
Semigroup variant	$\Sigma^+ \quad (aba, 3)$
Monoid variant	$\Sigma^* \quad (\varepsilon, 4)$

Semigroup Variant: Nonempty Words

Fixed: $\Sigma = \{a, b\}$ $\quad L = b^*(ab^*ab^*)^*$

Setting: adversarial arrival order, Σ^+ position content



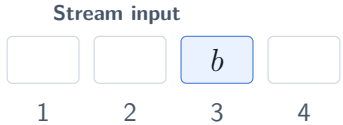
Flip if the new word has an odd number of a's.

Theorem: The semigroup variant for $b^*(ab^*ab^*)^*$ is in $O(1)$ space.

Semigroup Variant: Nonempty Words

Fixed: $\Sigma = \{a, b\}$ $\quad L = b^*(ab^*ab^*)^*$

Setting: adversarial arrival order, Σ^+ position content



new element: $(b, 3)$

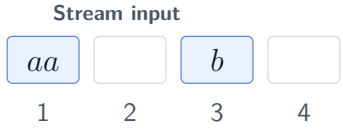
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Semigroup Variant: Nonempty Words

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Setting: adversarial arrival order, Σ^+ position content



new element: $(aa, 1)$

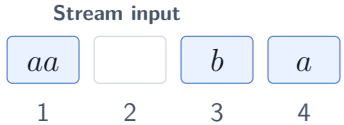
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Semigroup Variant: Nonempty Words

Fixed: $\Sigma = \{a, b\}$ $\quad L = b^*(ab^*ab^*)^*$

Setting: adversarial arrival order, Σ^+ position content



new element: $(a, 4)$

Flip if the new word has an odd number of a's.

Theorem: The semigroup variant for $b^*(ab^*ab^*)^*$ is in $O(1)$ space.

Semigroup Variant: Nonempty Words

Fixed: $\Sigma = \{a, b\}$ $\quad L = b^*(ab^*ab^*)^*$

Setting: adversarial arrival order, Σ^+ position content

Stream input



Memory: one parity bit

even

new element: $(a, 2)$

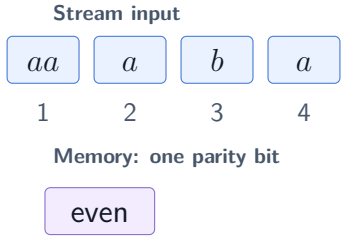
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Semigroup Variant: Nonempty Words

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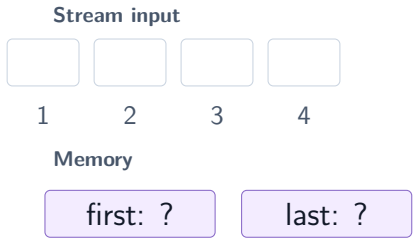
Decision: Accept iff final memory state is even

Theorem: The semigroup variant for $b^*(ab^*ab^*)^*$ is in $O(1)$ space.

Semigroup Variant: $a\Sigma^*b$ is in $O(1)$ Space

Fixed: $\Sigma = \{a, b\}$ $\quad L = a\Sigma^*b$

Setting: adversarial arrival order, Σ^+ position content

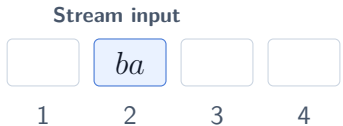


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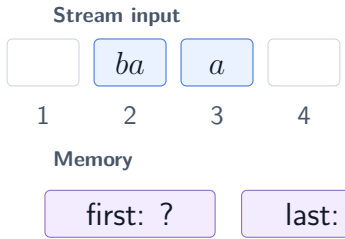
new element: $(ba, 2)$

Theorem: The semigroup variant for $a\Sigma^*b$ is in $O(1)$ space.

Semigroup Variant: $a\Sigma^*b$ is in $O(1)$ Space

Fixed: $\Sigma = \{a, b\}$ $\quad L = a\Sigma^*b$

Setting: adversarial arrival order, Σ^+ position content



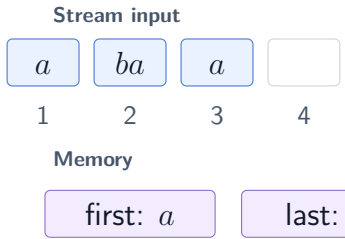
new element: $(a, 3)$

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Fixed: $\Sigma = \{a, b\}$ $\quad L = a\Sigma^*b$

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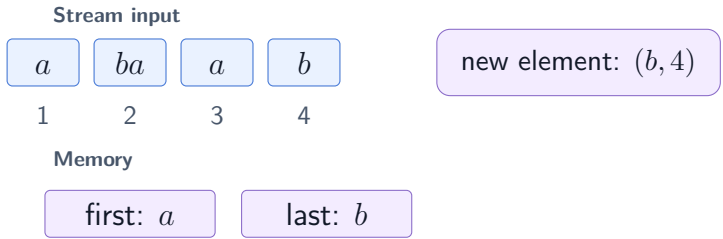


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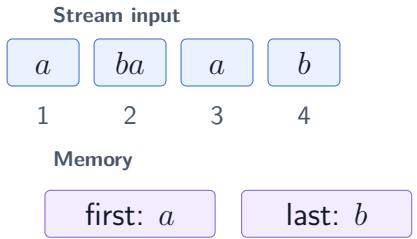


Theorem: The semigroup variant for $a\Sigma^*b$ is in $O(1)$ space.

Semigroup Variant: $a\Sigma^*b$ is in $O(1)$ Space

Fixed: $\Sigma = \{a, b\}$ $\quad L = a\Sigma^*b$

Setting: adversarial arrival order, Σ^+ position content



Decision: Accept iff first starts with a and last ends with b.

Theorem: The semigroup variant for $a\Sigma^*b$ is in $O(1)$ space.

Semigroup Variant: $O(1)$ Space

Definition

$L \subseteq \Sigma^+$ is in $\mathbf{LI}$ if there exists a constant k such that membership only depends on the first and last k letters:

$$\forall p, s \in \Sigma^k,\ \forall x \in \Sigma^*, \quad pxs \in L \Leftrightarrow ps \in L.$$

Theorem

A regular language has constant space data complexity for out-of-order membership in the semigroup variant if and only if it is in $\mathbf{LI} \vee \mathbf{Com}$.

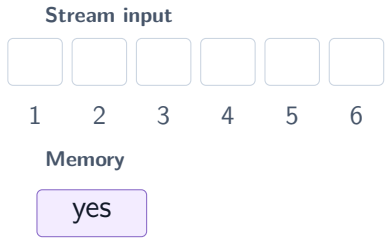
Problem variants

Out-of-order membership variant	position content
Original problem	$\Sigma \quad (a, 1)$
Semigroup variant	$\Sigma^+ \quad (aba, 3)$
Monoid variant	$\Sigma^* \quad (\varepsilon, 4)$

Original Problem: $(ab)^*$ is in $O(1)$ Space

Fixed: $\Sigma = \{a, b\}$ $L = (ab)^*$

Setting: adversarial arrival order, Σ position content



Theorem: Out-of-order membership in $(ab)^*$ is in $O(1)$ space.

Original Problem: $(ab)^*$ is in $O(1)$ Space

Fixed: $\Sigma = \{a, b\}$ $L = (ab)^*$

Setting: adversarial arrival order, Σ position content

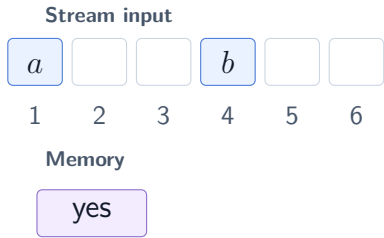


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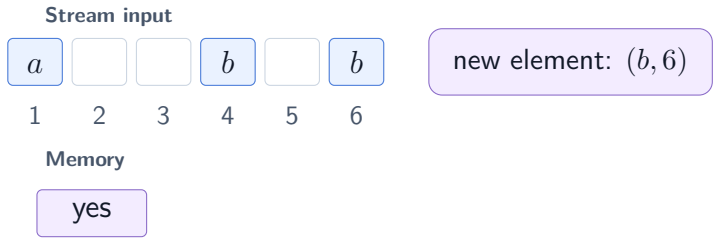


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Setting: adversarial arrival order, Σ position content

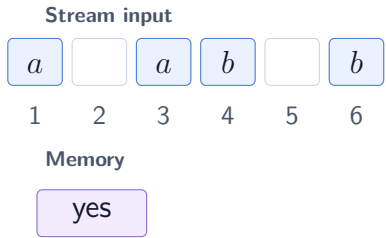


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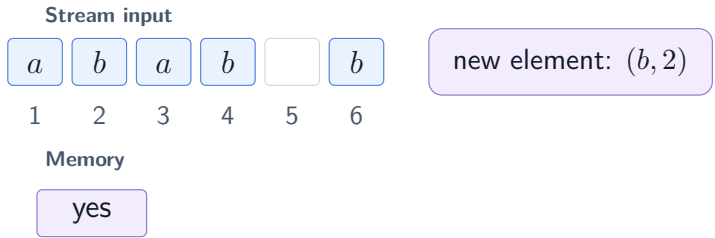


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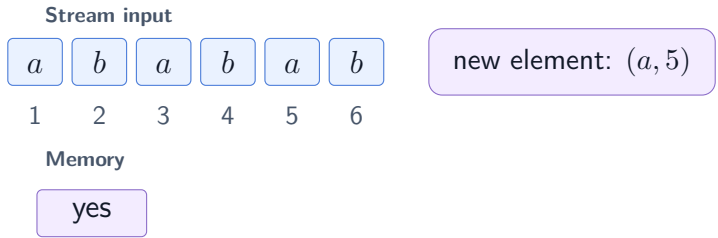


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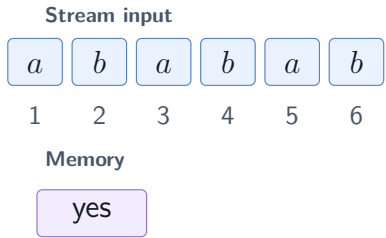


Theorem: Out-of-order membership in $(ab)^*$ is in $O(1)$ space.

Original Problem: $(ab)^*$ is in $O(1)$ Space

Fixed: $\Sigma = \{a, b\}$ $L = (ab)^*$

Setting: adversarial arrival order, Σ position content



Decision: Accept iff memory still says yes.

Theorem: Out-of-order membership in $(ab)^*$ is in $O(1)$ space.

Complexity Landscape

Space	Monoid	Semigroup	Language
$O(1)$	$\mathbf{Com}$ $b^*(ab^*ab^*)^*$	$\mathbf{LI} \vee \mathbf{Com}$ $a\Sigma^*b$	$(\mathbf{LI} \vee \mathbf{Com}) * \mathbf{Mod}$?
$O(\log n)$	$\mathbf{FL} \vee \mathbf{Com}$ a^*b^*	$\mathbf{MA} \vee \mathbf{LI} \vee \mathbf{Com}$? $a^*b^*a^*$	$(\mathbf{MA} \vee \mathbf{LI} \vee \mathbf{Com}) * \mathbf{Mod}$?
$O(\sqrt{n})$	–	$\mathbf{FLA} \vee \mathbf{Com}$? $\Sigma^*a\Sigma^*b\Sigma^*c\Sigma^*$	$(\mathbf{FLA} \vee \mathbf{Com}) * \mathbf{Mod}$?
$O(n)$	$\mathbf{all}$ $\Sigma^*aa\Sigma^*$	$\mathbf{all}$ $(ab)^*$	$\mathbf{all}$

Open Problem: $\Sigma^* a \Sigma^* b \Sigma^* c \Sigma^*$ - abc subword

Fixed: $\Sigma = \{a, b, c\}$ $\quad L = \Sigma^* a \Sigma^* b \Sigma^* c \Sigma^*$

Setting: adversarial arrival order, Σ^+ position content

Known bounds

- $\Omega(\log n)$ lower bound
- $O(\sqrt{n})$ upper bound (not published)