Edge-Minimum Walk of Modular Length in Polynomial Time

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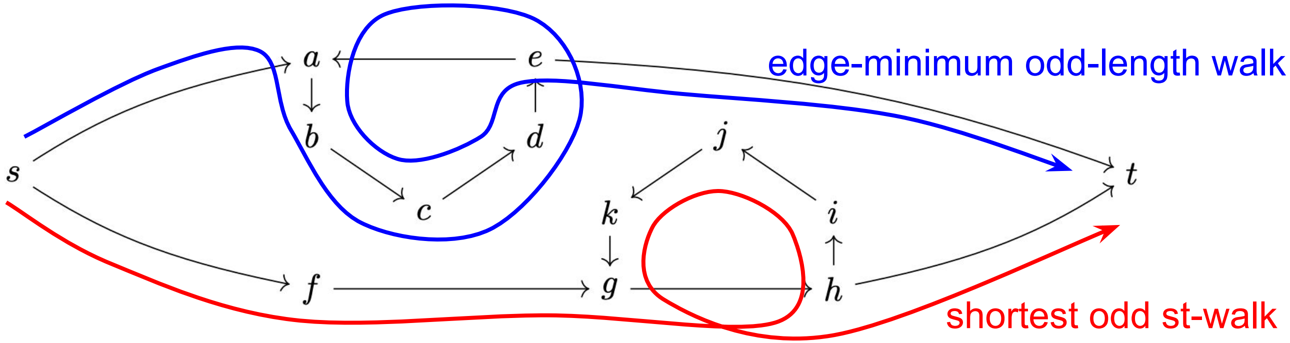
Question: Given an unweighted directed graph G with terminals s,t, can we efficiently find an **odd-length st-walk** with the **minimum number of distinct edges**?

Not the same as:

- shortest odd *st-path* problem on a directed graph (NP-hard)
- shortest odd *st-walk* problem on a directed graph (poly time)



General Problem: Edge-Minimum Walk of Modular Length

Problem: Given an unweighted directed graph G with terminals s,t, find an **st-walk of length r mod q** which is **edge-minimum** (minimum number of distinct edges).

Main result: Polynomial-time algorithm for constant q.

Specifically: $n^{O(\log q)} \cdot 2^{O(q \log^2 q)}$

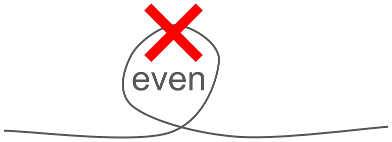
Generalizations:

- weighted versions
- a multi-terminal version of the problem that also generalizes the Directed Steiner Network problem.

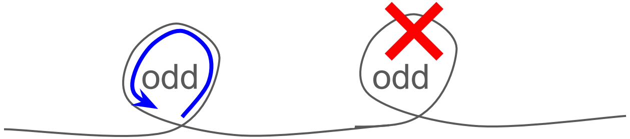
Proof Idea for Special Case: Odd-Length Walk

Goal: Find an edge-minimum odd-length st-walk w in a directed graph.

Obs 1. wlog w contains **no even cycles**.



Obs 2. wlog w does **not** contain **two odd cycles** in succession.



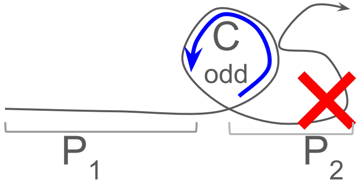
Conclusion: w is either a (1) simple path, or

(2) simple path P_1 + odd cycle C + simple path P_2.

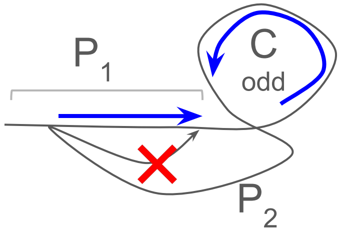
Proof Idea for Special Case: Odd-Length Walk

Goal: Find an edge-minimum odd-length st-walk w in a directed graph.

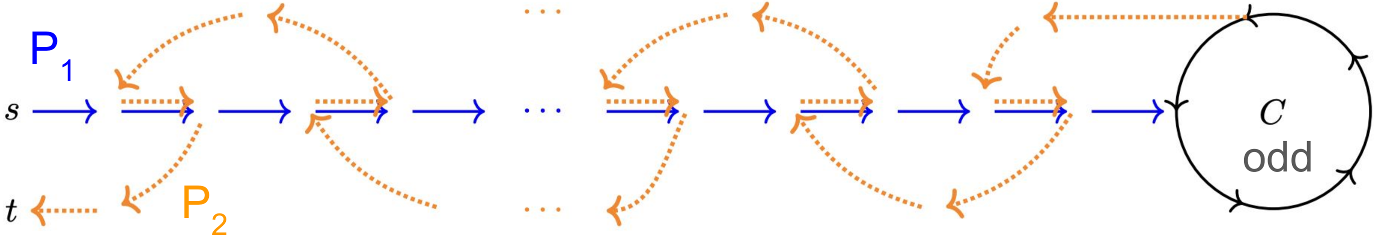
Obs 3. wlog P_2 doesn't re-intersect C.



Obs 4. wlog P_2 doesn't contain a "parallel path" with P_1.



Conclusion:



Proof Idea for Special Case: Odd-Length Walk

Goal: Find an edge-minimum odd-length st-walk w in a directed graph.

Idea from [Feldman/Ruhl '06, Feldmann/Marx '23] for Directed Steiner Network:

Bound the undirected **cutwidth** of the solution,

then use dynamic programming.

A graph has **cutwidth** $\leq c$ if there's an ordering of the vertices so that for every i, there are $\leq c$ edges with one endpoint $< i$ and one endpoint $\geq i$.

Extending to General r mod q

If w has a cycle of a length co-prime to q, argument is similar to odd case.

Issue: Could be many different cycles with different remainders mod q that intersect in intricate ways.

Question: Could a high-cutwidth (e.g. grid-like) structure emerge?

Answer: No

Structure of proof:

1. When w witnesses certain intersection patterns, more remainders become "achievable".
2. Bound total number of certain intersection patterns.
3. Bound cutwidth.

Open Problems

1. Improve running time below $n^{O(\log q)} \cdot 2^{O(q \log^2 q)}$.
2. Unexplored problem (we think): Given a number ℓ, find an edge-minimum st-walk of length exactly ℓ, faster than the trivial $O(n^{\ell})$ time.
3. Intermediate problems between shortest walk and edge-minimum walk.
4. Generalizations related to *regular path queries.*

Thank you!