Topological Sorting under Regular Constraints

Antoine Amarilli[1], **Charles Paperman**[2]

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[1]Télécom ParisTech

[2]Université de Lille

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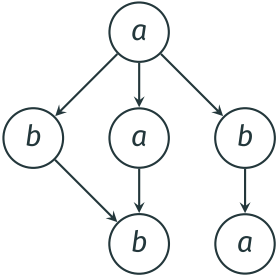

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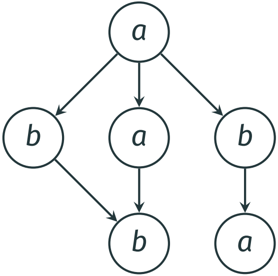

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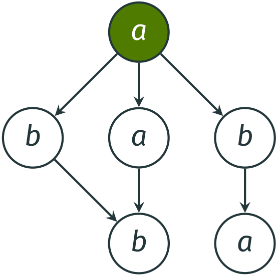

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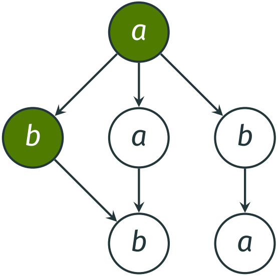

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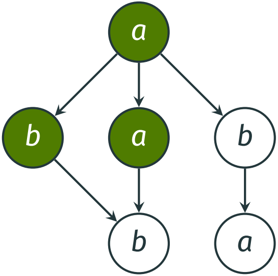

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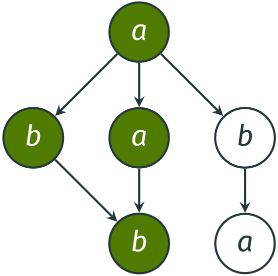

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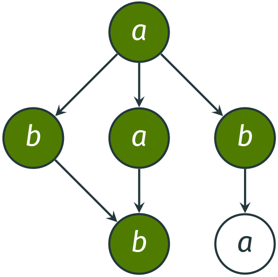

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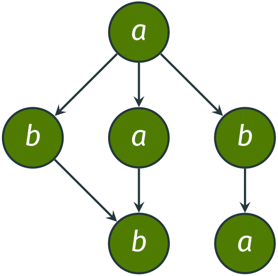

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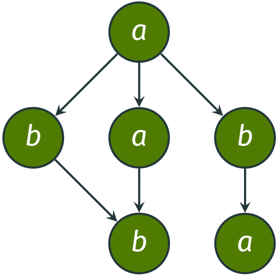

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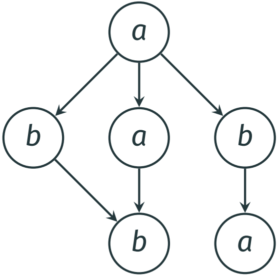

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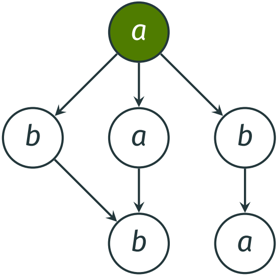

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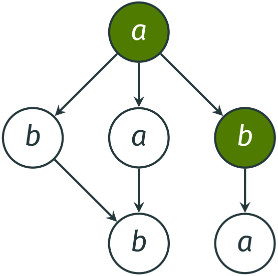

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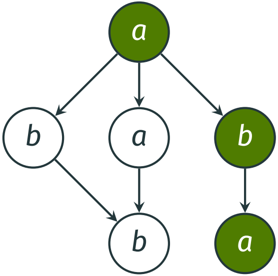

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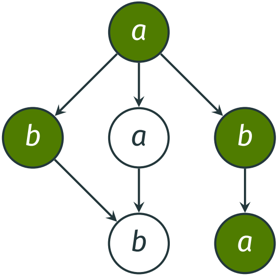

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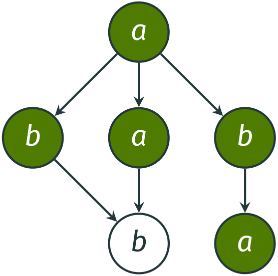

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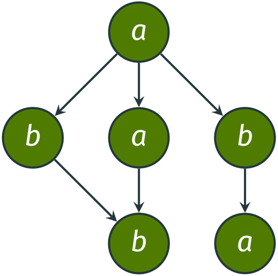

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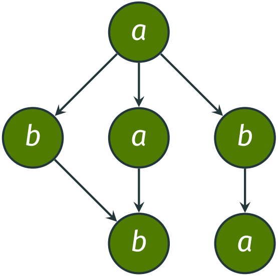

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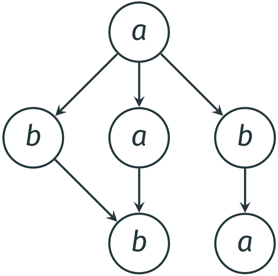

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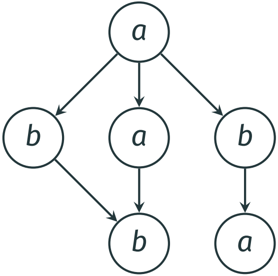
- **Question:** when is this problem tractable?



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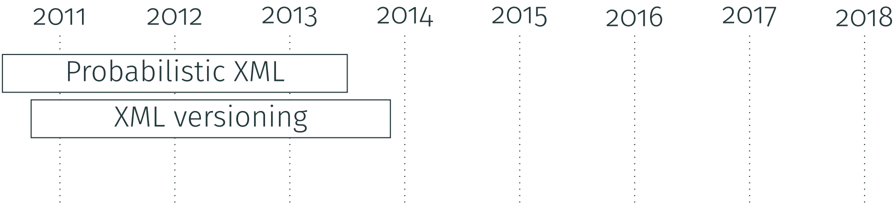

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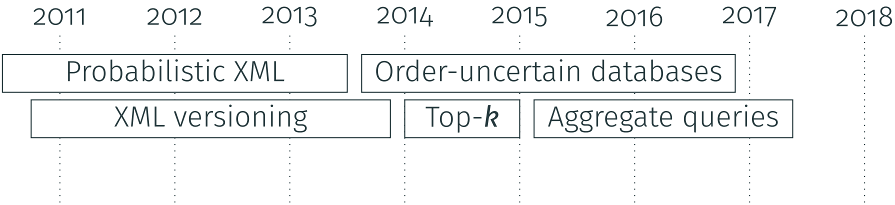

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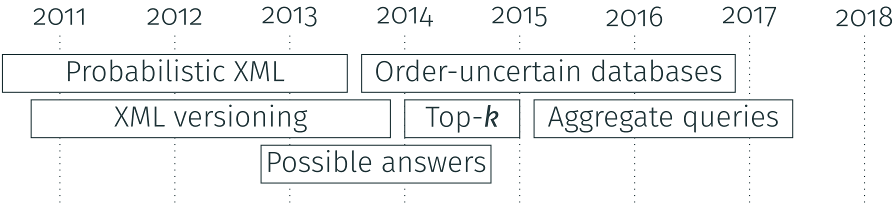

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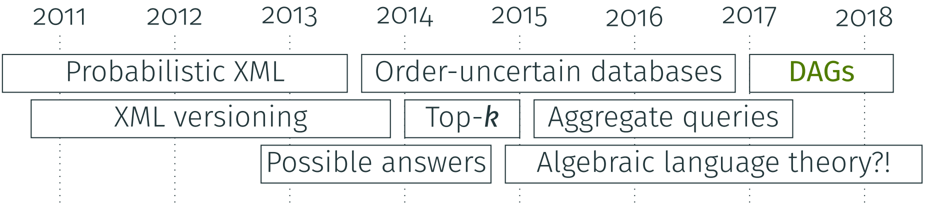

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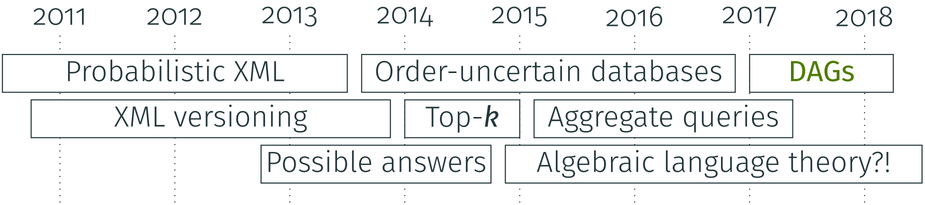

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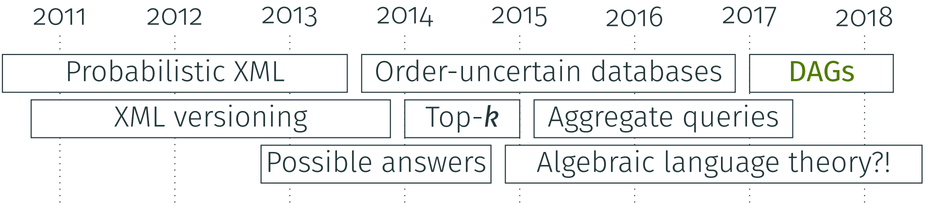

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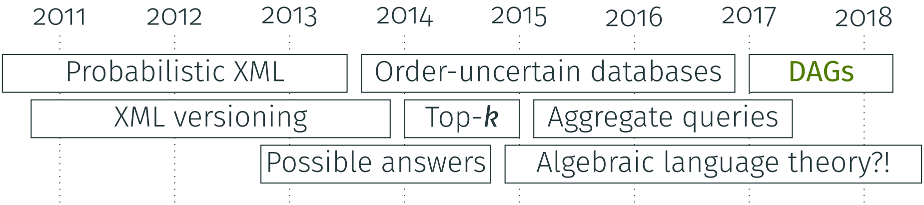

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→ Computational biology! → Blockchain!

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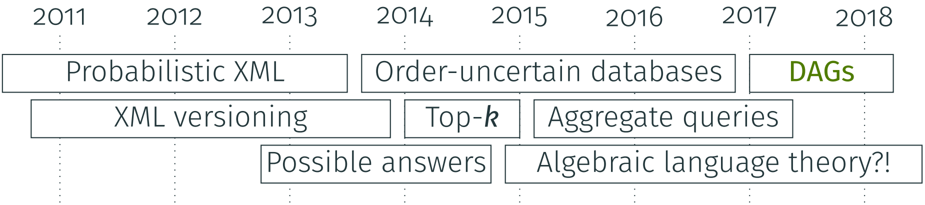

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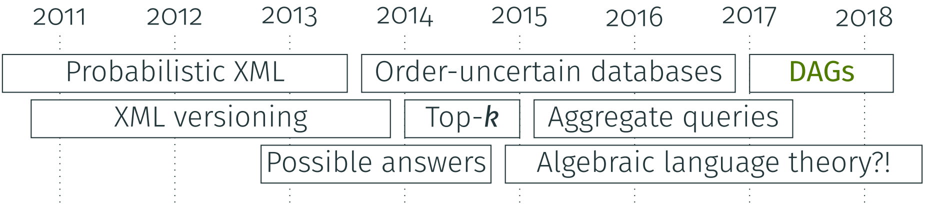

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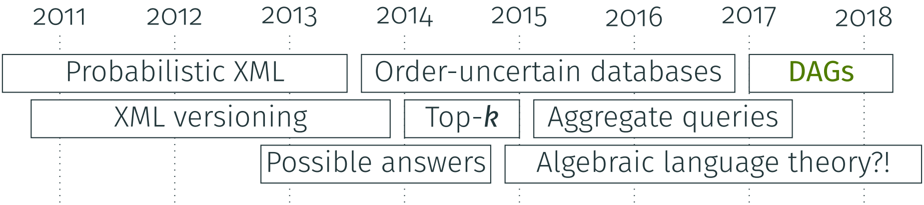

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→ **Natural** problem and apparently **nothing** was known about it!

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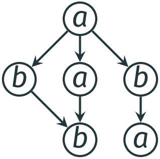

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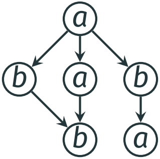

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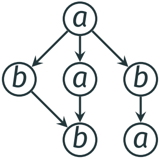

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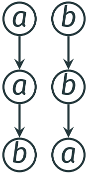

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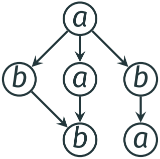

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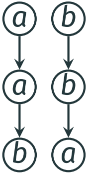

→ **Question:** What is the **complexity** of $\mathsf{CTS}(L)$ and $\mathsf{CSh}(L)$, depending on the fixed language L?

Dichotomy

*For every **regular language L**, exactly one of the following holds:*

- *L has [some nice property] and $\mathsf{CTS}(L)$ is **in NL***
- *L has [some nasty property] and $\mathsf{CTS}(L)$ is **NP-hard***

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$\rightarrow$ Very mysterious landscape! (to us)

Hardness Results

Existing Hardness Result

JOURNAL OF COMPUTER AND SYSTEM SCIENCES **28**, 345–358 (1984)

On the Complexity of Iterated Shuffle*

MANFRED K. WARMUTH† AND DAVID HAUSSLER‡

Department of Computer Science,
University of Colorado, Boulder, Colorado 80309

It is demonstrated that the following problems are *NP* complete:

(1) Given words w and $w_1, w_2, \ldots, w_n$, is w in the shuffle of $w_1, w_2, \ldots, w_n$?

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$\rightarrow$ Does not directly apply for us, because we **fix** the target language

Hardness for CTS

- We can reduce their problem to CSh for the language $(aA + bB)^*$
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- Custom reduction technique to show NP-hardness

The reduction technique

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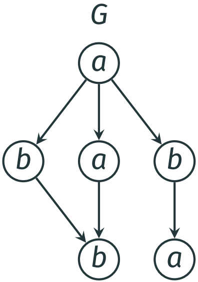

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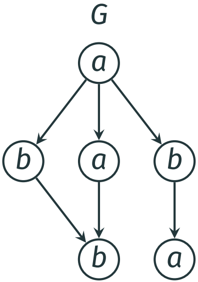

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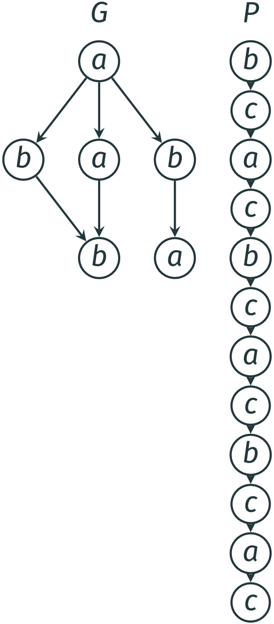

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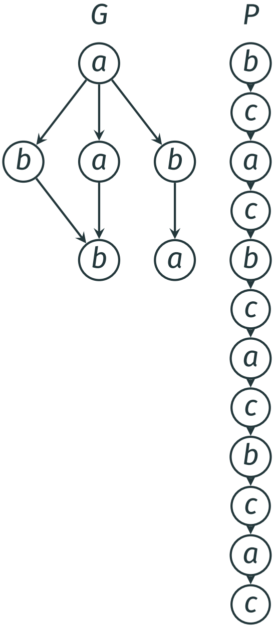

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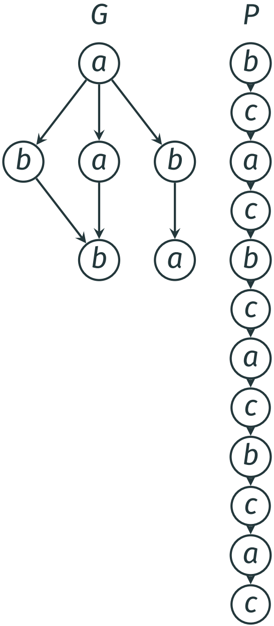

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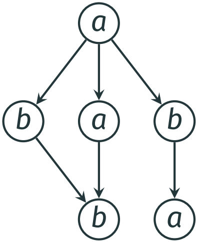

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Theorem

If L shuffle-reduces to L' then:

- $\mathsf{CSh}(L)$ *reduces in PTIME to* $\mathsf{CSh}(L')$
- $\mathsf{CTS}(L)$ *reduces in PTIME to* $\mathsf{CTS}(L')$

Other hard languages

- The reduction shows hardness for:
  - $(ab + b)^*$ (also simpler argument)
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  - Idea: reason about consumption rates of letters?
  - Not even complete for F^* languages, as $(aa + bb)^*$ is NP-hard

Tractability Results

Tractability for Monomials

- **Monomial:** language of the form $A_1^* \, a_1 \, A_2^* \, a_2 \, \cdots \, A_n^* \, a_n \, A_{n+1}^*$ where $a_1, \ldots, a_n \in \Sigma$ and $A_1, \ldots, A_{n+1} \subseteq \Sigma$
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Theorem

For any union of monomials L*, the problem* $\mathsf{CTS}(L)$ *is* ***in NL***

Proof Idea for Monomials

- Tractable languages are clearly closed under union
  so it suffices to consider a monomial: $A_1^* \, a_1 \, A_2^* \, a_2 \, \cdots \, A_n^* \, a_n \, A_{n+1}^*$
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  - Inductively solve the problem for these vertices and $A_1^* \, a_1 \, A_2^* \, a_2 \, \cdots \, A_n^*$

The Algebraic Approach

Can we just study algebraically the tractable languages?

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Can we just study algebraically the tractable languages? Not really...

- Not closed under intersection
- Not closed under complement
- Not closed under inverse morphism
- Not closed under concatenation (not in paper, only known for CTS)
- For CSh: not closed under quotient

¯_(ツ)_/¯

Side Remark: CTS and CSh are Different

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Hence, some languages are tractable for CSh and hard for CTS

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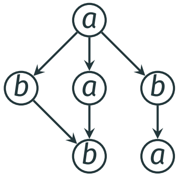

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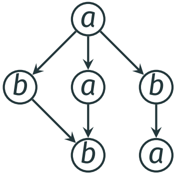

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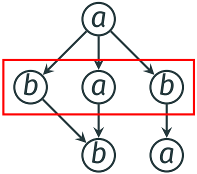

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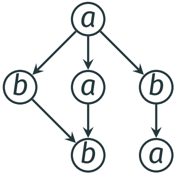

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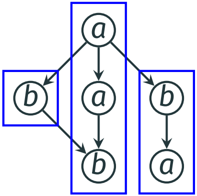

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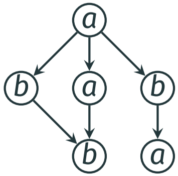

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$\rightarrow$ Unknown for $L + \Sigma^* a^k \Sigma^*$ with arbitrary L and $k > 2$! ¯_(ツ)_/¯

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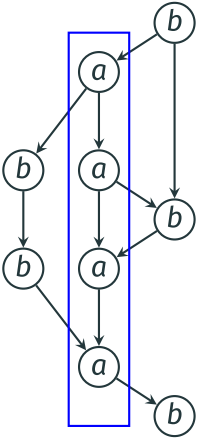

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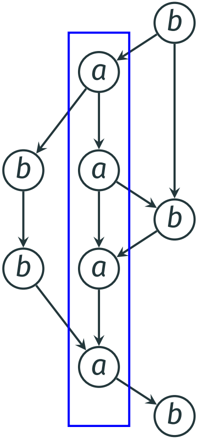

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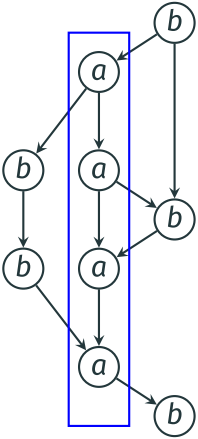

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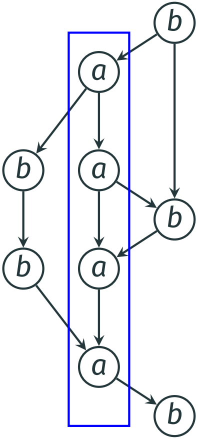

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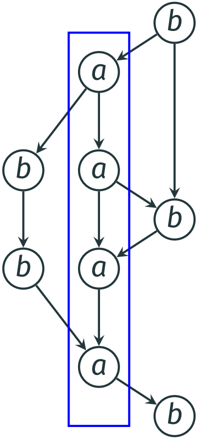
An Annoying Open Problem



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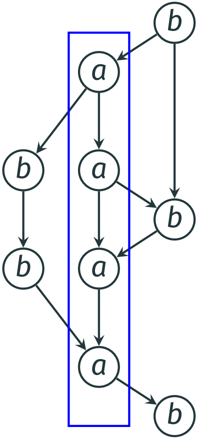

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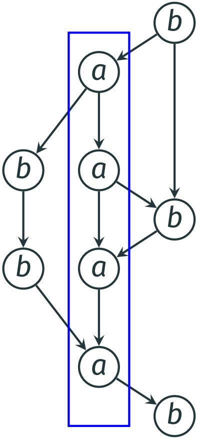

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Open problem

Fix $\Sigma = \{a, b\}$ and an arbitrary regular language L. Given a DAG without two incomparable a's, can you solve CTS(L)?

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Theorem

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→ Only for CSh; complexity for CTS is unknown! ¯_(ツ)_/¯

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  - When doing the NL algorithm on rare letters, constant bound on the number of frequent letter insertions

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- $(caa)^*d(cbb)^*d\Sigma^* + \Sigma^*cc\Sigma^*$ is in NL for CSh but NP-hard for CTS
  - Tractability argument: another ad hoc greedy algorithm
  - Hardness argument: from k-clique encoded to a bipartite graph

A Kind of Dichotomy

Prelude to the Kind of Dichotomy

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- We were aiming for a dichotomy, but... ¯_(ツ)_/¯
- Let's try to make the problem simpler
- Idea: If we don't fix a target language but a language "family" then we can hope for a coarser dichotomy
- We can restrict to "families" closed under algebraic operations and go back to the algebraic approach

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- Fix a **semiautomaton** $S = (Q, \Sigma, \delta)$ with Q the set of **states**, with Σ a **finite alphabet**, and with δ the **transitions**.

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- Semiautomaton Constrained topological sort problem $\mathsf{CTS}(S)$:
  - Input:
    - a DAG G with vertices labeled by letters of Σ,
    - a specification of S, i.e., $\{(i_1, F_1), \ldots, (i_k, F_k)\}$ with $(i_j, F_j) \in Q \times 2^Q$
  - Output: is there a topological sort of G such that the sequence of vertex labels is accepted by the automaton $(Q, \Sigma, \delta, i_j, F_j)$ for all $1 \leq j \leq k$

A Kind of Dichotomy (2)

Theorem

For every *semiautomaton S, exactly one of the following holds:*

- *The transition semigroup of* S *belongs to* ... *and* $\mathsf{CTS}(S)$ *is in NL*
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Theorem

For every ~~*counterfree*~~ *semiautomaton S, exactly one of the following holds:*

- *The transition semigroup of* S *belongs to DO and* CSh(*S*) *is in NL*
- *The transition semigroup of* S *is not in DS and* CSh(*S*) *is NP-hard*

- *DA* is a classic variety of semigroups
- Counterfree is equivalent to being first-order definable and “not containing any groups”
- *DO*, *DS* are much less well understood varieties of semigroups

Conclusion

Summary and Future Work

Language	**CSh** (shuffle)	**CTS** (top. sort)
$(ab)^*$, u^* with different letters	NP-hard	NP-hard

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$L + \Sigma^*(a^k + b^k)\Sigma^*$	in NL	in NL
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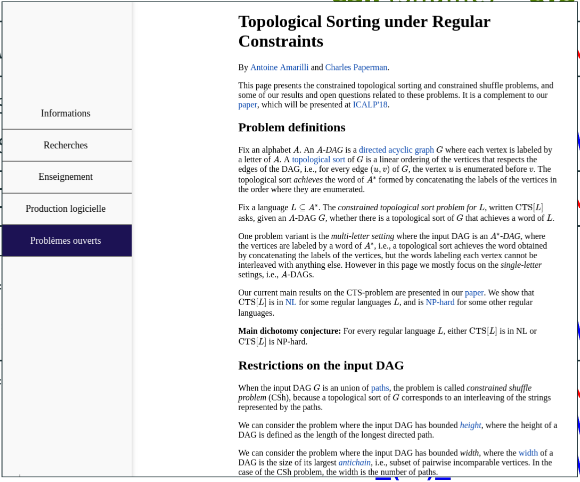

Informations

Recherches

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Production logicielle

Problèmes ouverts

Topological Sorting under Regular Constraints

By Antoine Amarilli and Charles Paperman.

This page presents the constrained topological sorting and constrained shuffle problems, and some of our results and open questions related to these problems. It is a complement to our paper, which will be presented at ICALP'18.

Problem definitions

Fix an alphabet A. An *A-DAG* is a directed acyclic graph G where each vertex is labeled by a letter of A. A topological sort of G is a linear ordering of the vertices that respects the edges of the DAG, i.e., for every edge (u, v) of G, the vertex u is enumerated before v. The topological sort *achieves* the word of A^* formed by concatenating the labels of the vertices in the order where they are enumerated.

Fix a language $L \subseteq A^*$. The *constrained topological sort problem for* L, written $\mathrm{CTS}[L]$ asks, given an A-DAG G, whether there is a topological sort of G that achieves a word of L.

One problem variant is the *multi-letter setting* where the input DAG is an *A^*-DAG*, where the vertices are labeled by a word of A^*, i.e., a topological sort achieves the word obtained by concatenating the labels of the vertices, but the words labeling each vertex cannot be interleaved with anything else. However in this page we mostly focus on the *single-letter* setings, i.e., A-DAGs.

Our current main results on the CTS-problem are presented in our paper. We show that $\mathrm{CTS}[L]$ is in NL for some regular languages L, and is NP-hard for some other regular languages.

Main dichotomy conjecture: For every regular language L, either $\mathrm{CTS}[L]$ is in NL or $\mathrm{CTS}[L]$ is NP-hard.

Restrictions on the input DAG

When the input DAG G is an union of paths, the problem is called *constrained shuffle problem* (CSh), because a topological sort of G corresponds to an interleaving of the strings represented by the paths.

We can consider the problem where the input DAG has bounded *height*, where the height of a DAG is defined as the length of the longest directed path.

We can consider the problem where the input DAG has bounded *width*, where the width of a DAG is the size of its largest *antichain*, i.e., subset of pairwise incomparable vertices. In the case of the CSh problem, the width is the number of paths.

Summary and Future Work

Language		CTS (top. sort)
$(ab)^*$, u^* with		NP-hard
Monomials $A_1^* a$ Groups, distric		in NL ¯_(ツ)_/¯
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Thanks for your attention!

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