Dynamic Membership for Regular Tree Languages

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Joint work with Corentin Barloy, Pawel Gawrychowski, Louis Jachiet, Charles Paperman

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What is the complexity of this problem? can we do better than $O(n)$ per update?

Dynamic membership for words: Results (actually for monoids)

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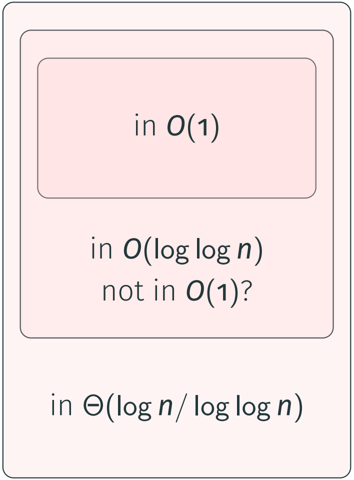

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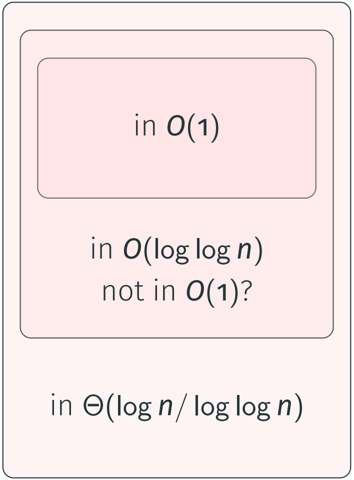

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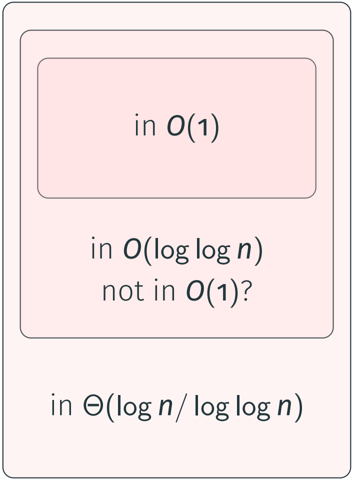

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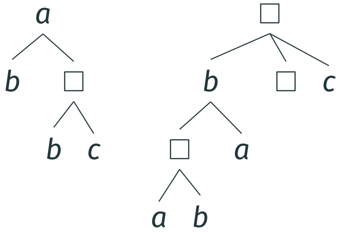

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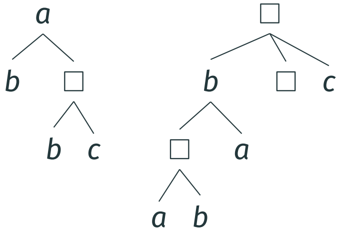
Innocent question: **"What about regular tree languages"?**

- Rooted, ordered, **unranked forests**
- Label on **nodes** from finite alphabet

e.g., $\Sigma = \{a, b, c, d, \square\}$



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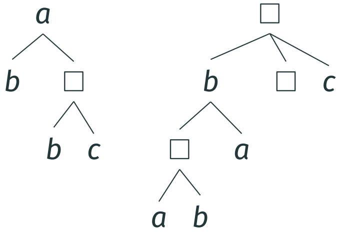

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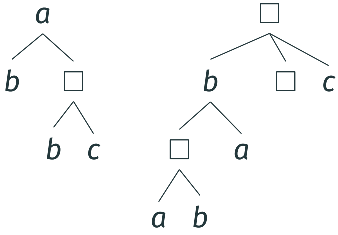

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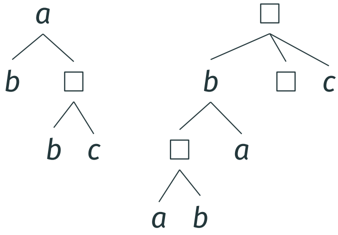

Some big **simplifying assumptions**:

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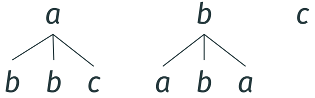
- **Special neutral label** $\square$ means "replace the node by the list of its children" (avoids "local information")



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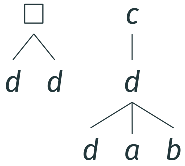

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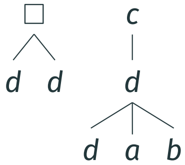

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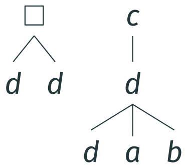

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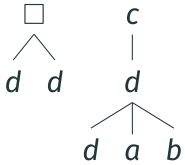

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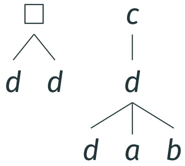

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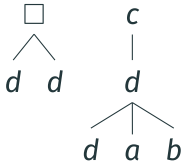

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[See PhD of Corentin Barloy, Section 7.2]

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Existential marked ancestor problem: "there is one b with an a ancestor"

This (aperiodic) problem has an unconditional $\Omega(\log n / \log \log n)$ lower bound!
(from Alstrup et al., FOCS'98)

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But we are still missing a characterization of the $O(\log \log n)$ boundary...

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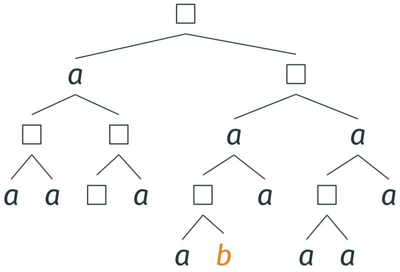

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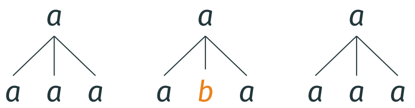

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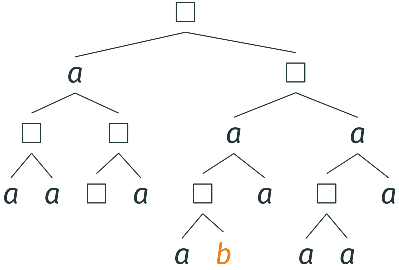

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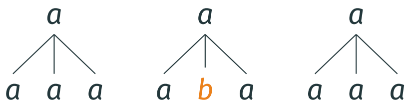

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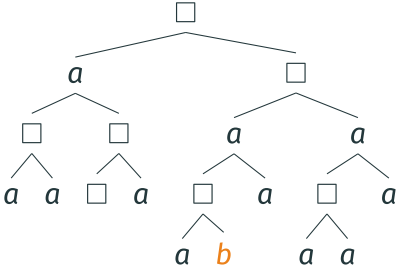

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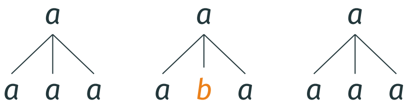

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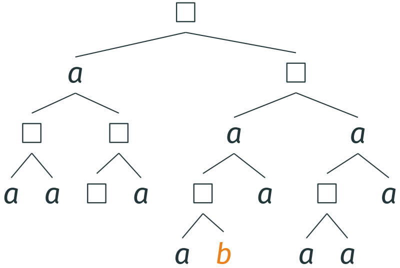

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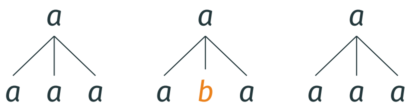

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What is the complexity?? no known reduction from existential marked ancestor

References i

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