Smallest Witnesses for RPQs

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Joint work with Benoît Groz, Nicole Wein

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[1]Inria Lille

Smallest Witness for Boolean Queries

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- For a UCQ Q the task is always **PTIME** (smallest witnesses have size $\leq |Q|$)

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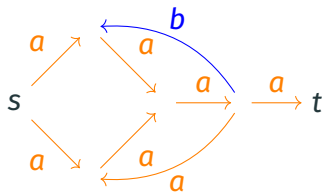

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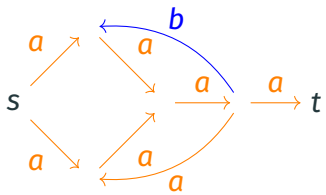

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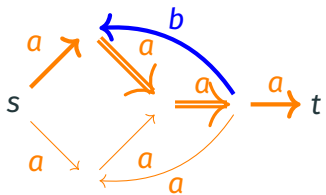

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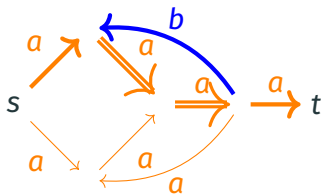

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- We can check in PTIME whether $D \models \mathrm{RPQ}_L(s, t)$
  - → Evaluate in Datalog, or do product construction of D with automaton for L

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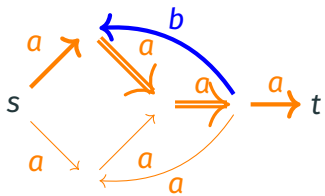

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  → Compute the shortest path from s to t and check that it has $\leq k$ edges

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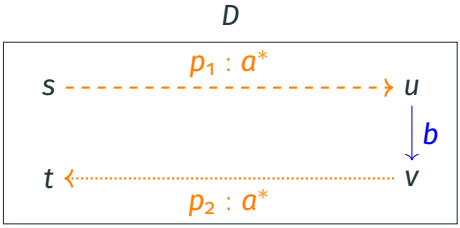

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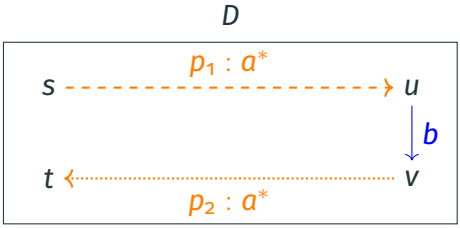

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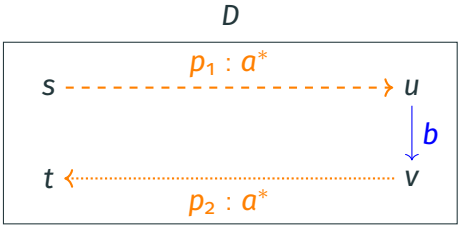

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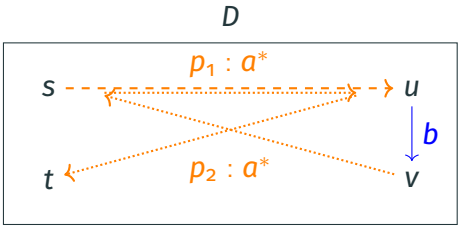

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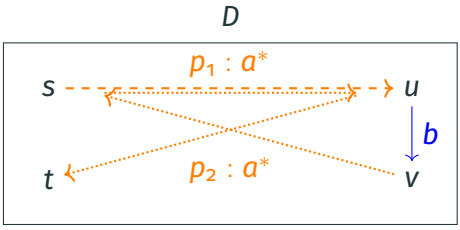

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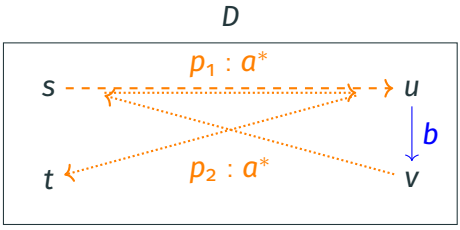

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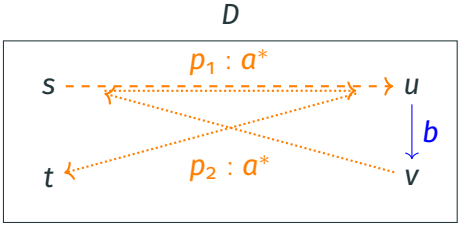

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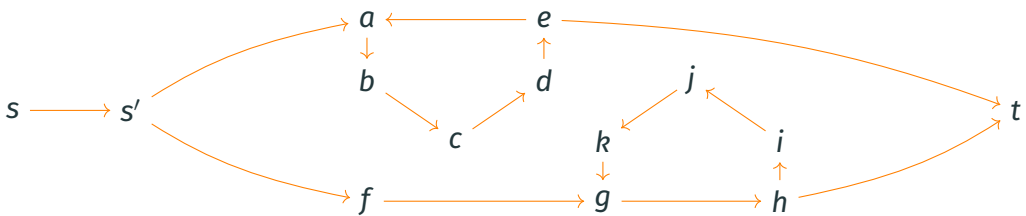

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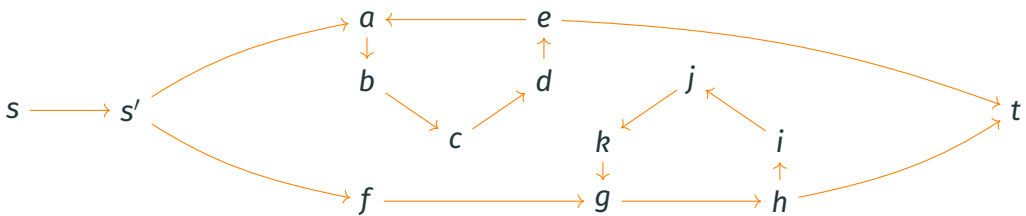

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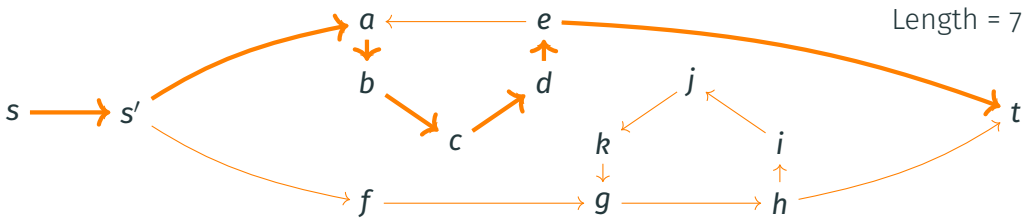

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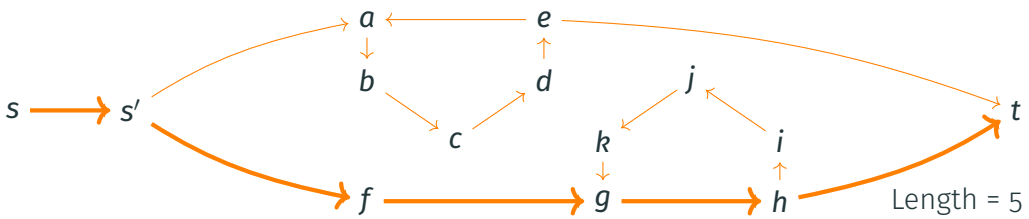

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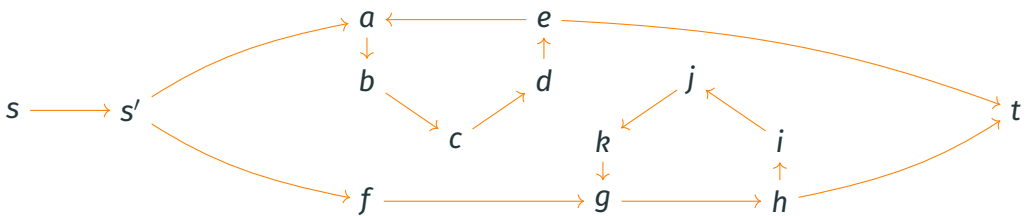

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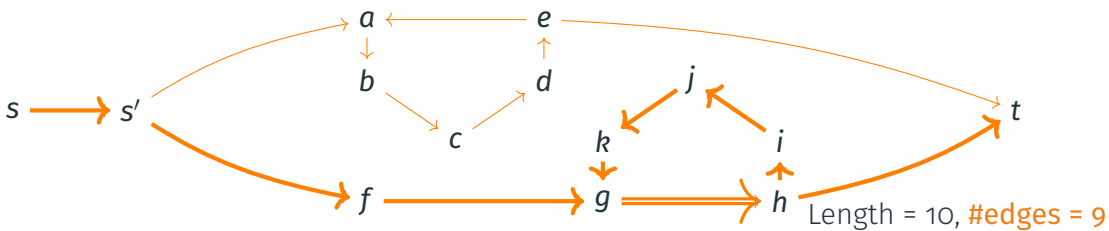

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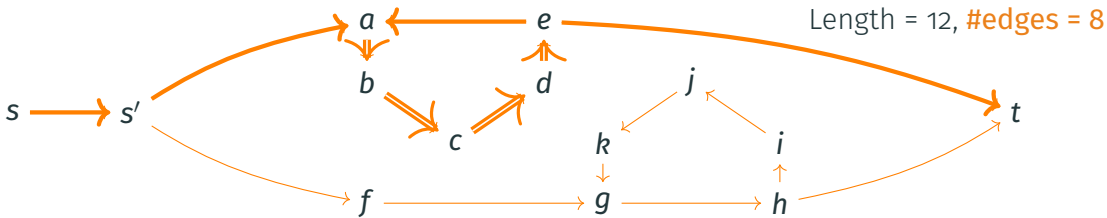

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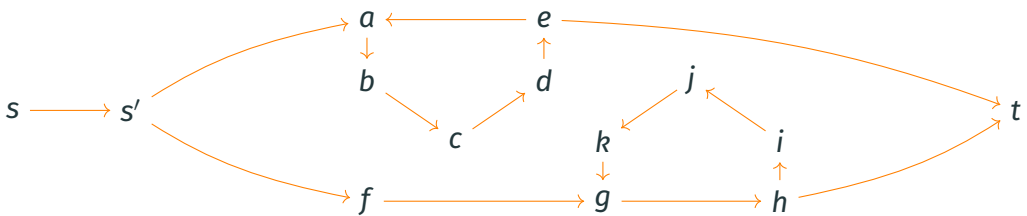

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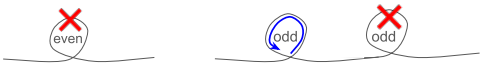

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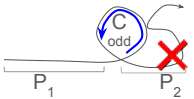

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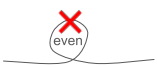

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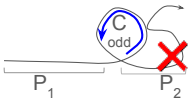

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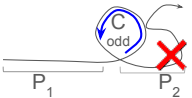

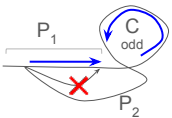

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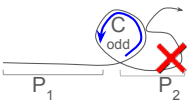

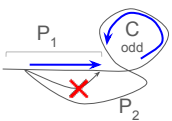

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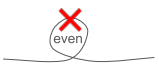

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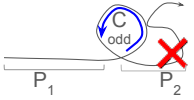

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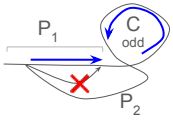

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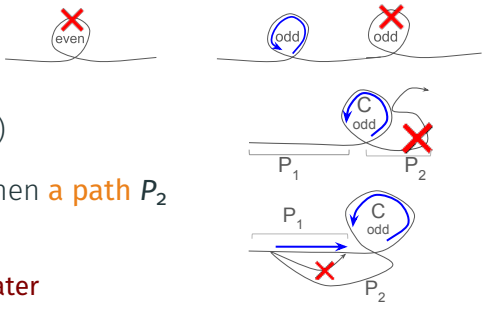

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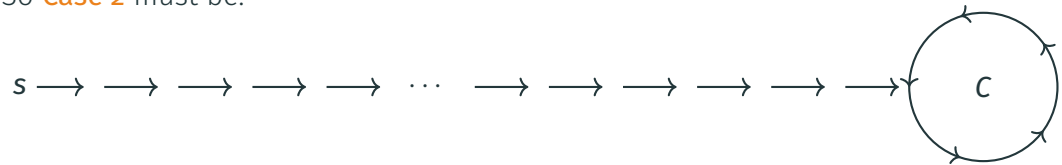

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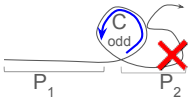

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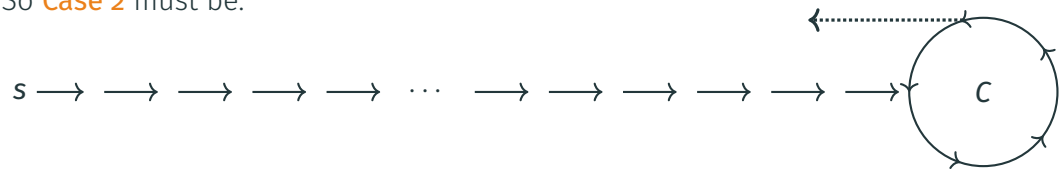

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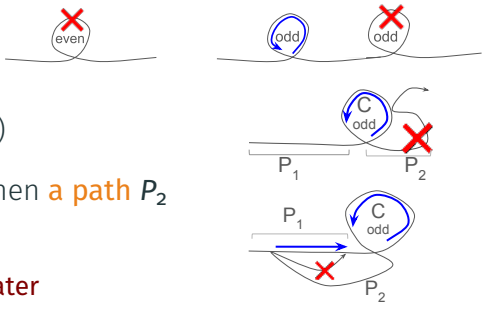

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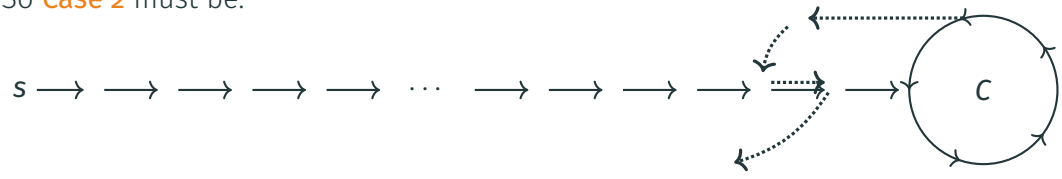

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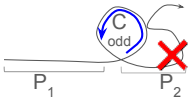

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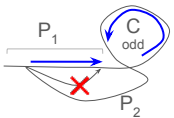

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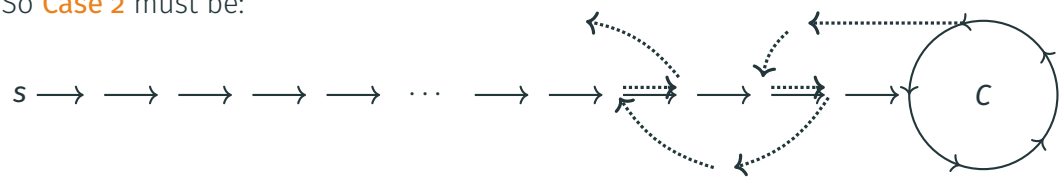

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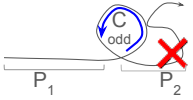

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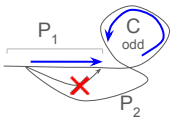

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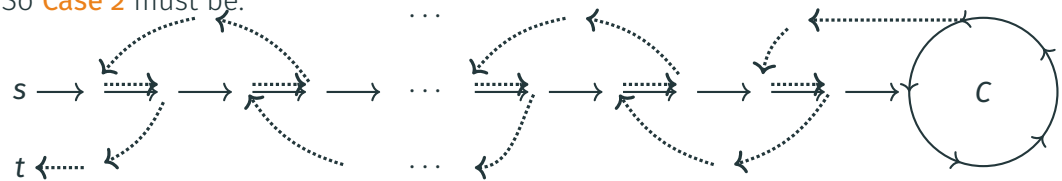

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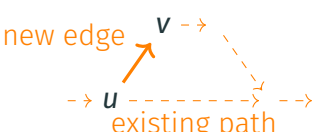

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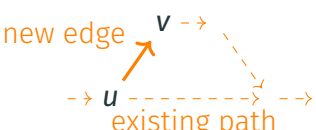

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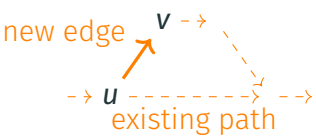

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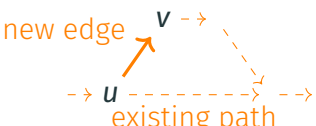

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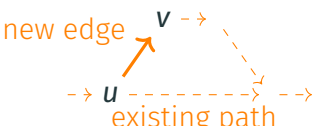

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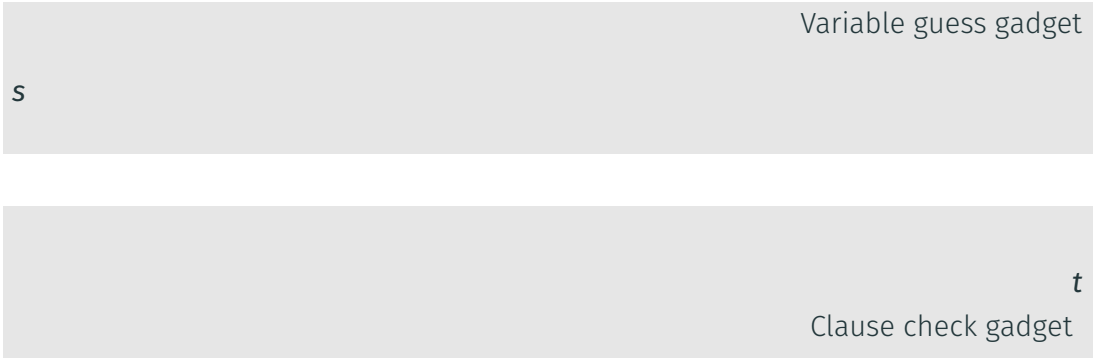

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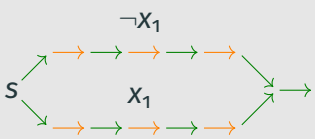

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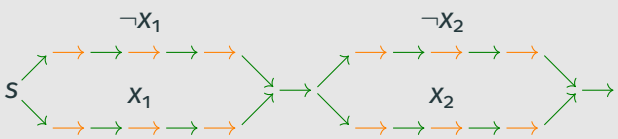

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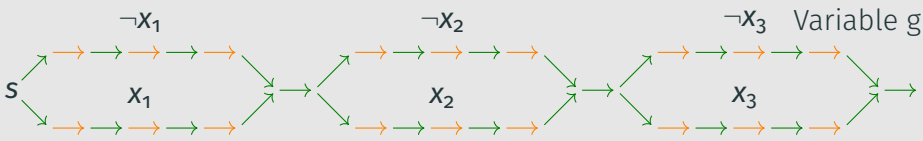

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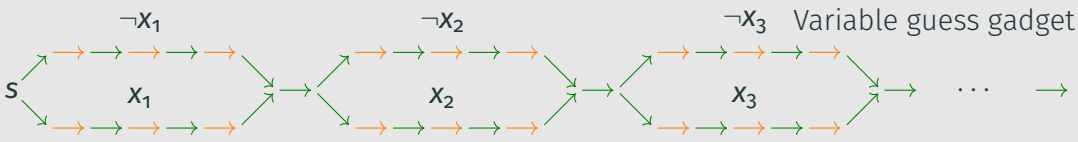

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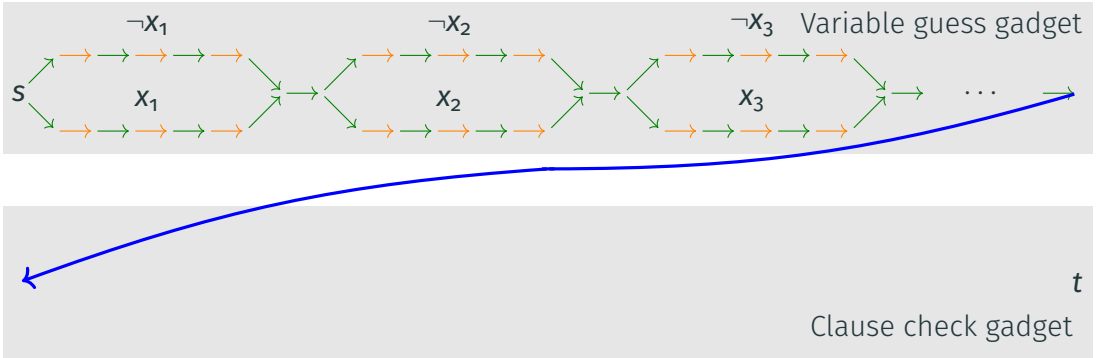

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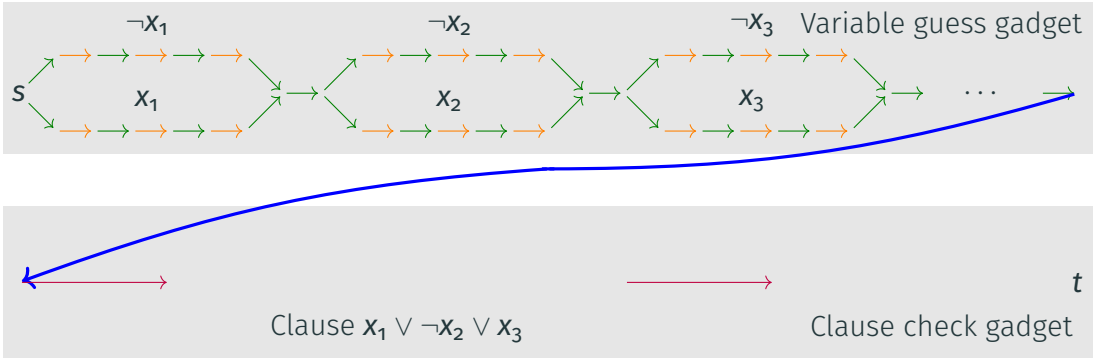

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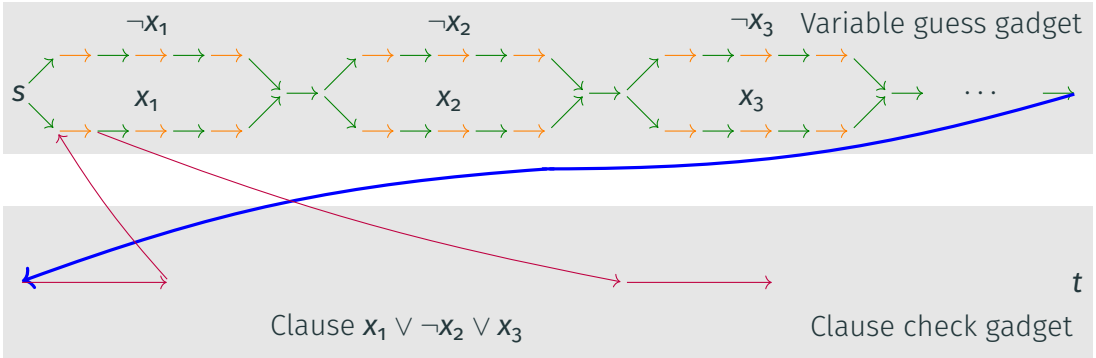

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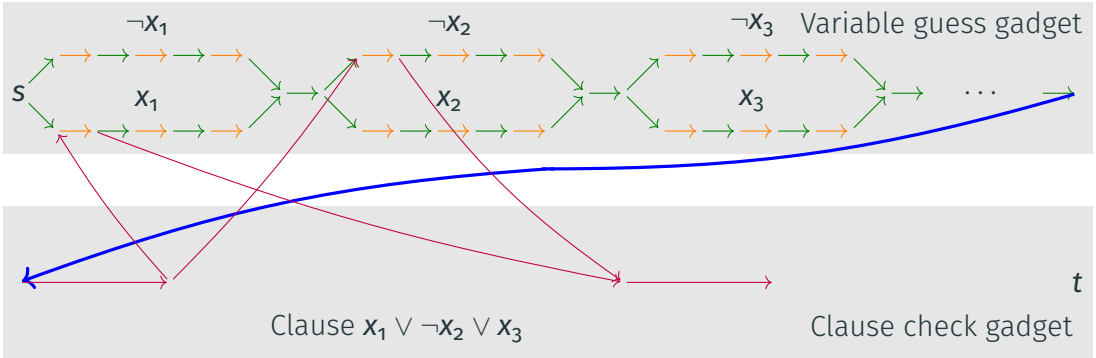

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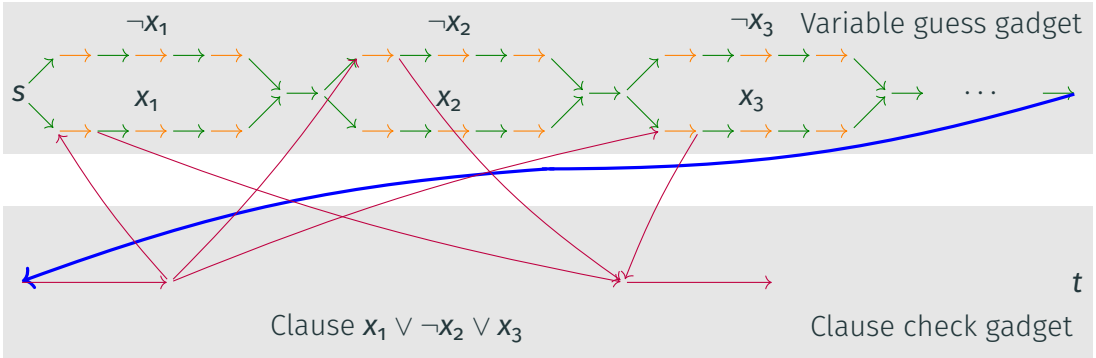

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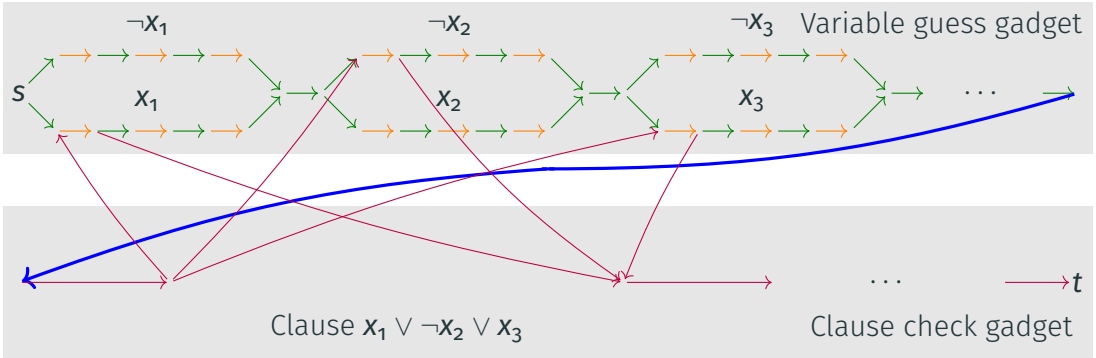

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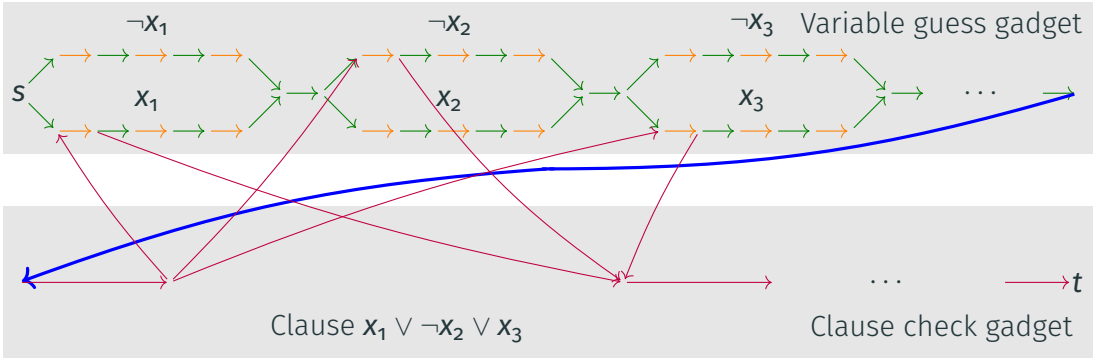

Minimizing #distinct edges forces that all a-edges in the clause checks are revisits

Open Question 1: Dichotomy on Smallest Witness for RPQs?

Conjecture (with Benoît Groz)

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Reminiscent of CSP/VSCP but no clear connection:

> SW_L: "For fixed RHS structure H_L, given a LHS structure G, find a minimal substructure of G that does not have a homomorphism to H_L"

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Thanks for your attention!

Acknowledgements

Slide 7: Thanks to Nicole Wein for preparing some of the drawings.

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