Smallest Witnesses for RPQs

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Joint work with Benoît Groz, Nicole Wein
Thanks to Benoît Groz and Nicole Wein for part of the slides

April 1st, 2026

[1]Inria Lille

Outline

Smallest witnesses, RPQs

Tractable cases for SW

Hardness results

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Smallest Witness for Boolean Queries

- Fix a Boolean query Q

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- For a UCQ Q the task is always **PTIME** (smallest witnesses have size $\leq |Q|$)

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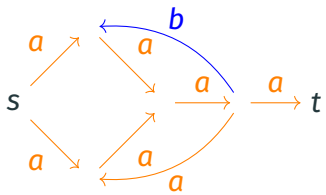

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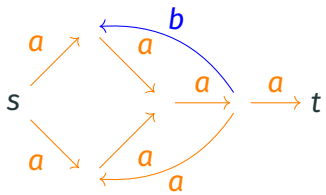

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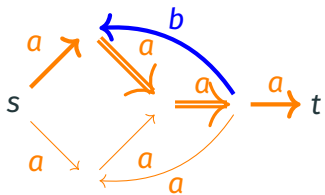

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- We can check in PTIME whether $D \models \mathrm{RPQ}_L(s, t)$
  - → Evaluate in Datalog, or do product construction of D with automaton for L

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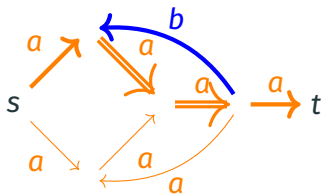

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  → Compute the shortest path from s to t and check that it has $\leq k$ edges

Tractable cases for SW

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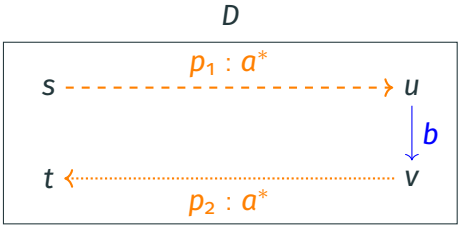

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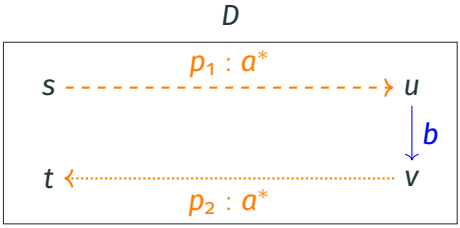

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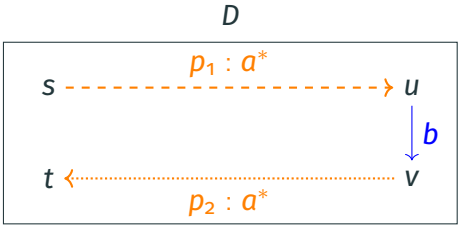

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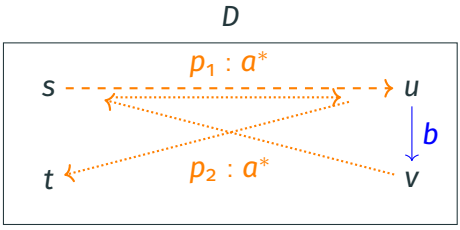

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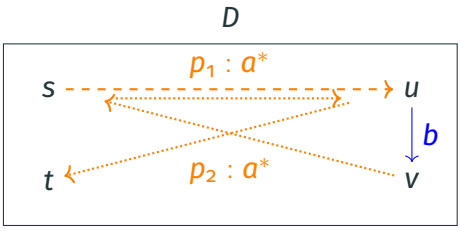

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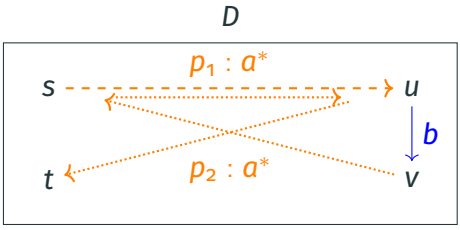

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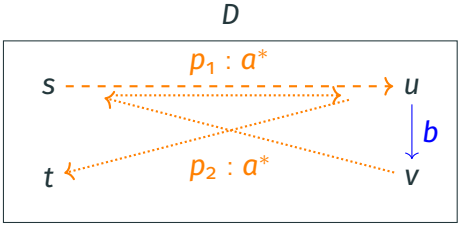

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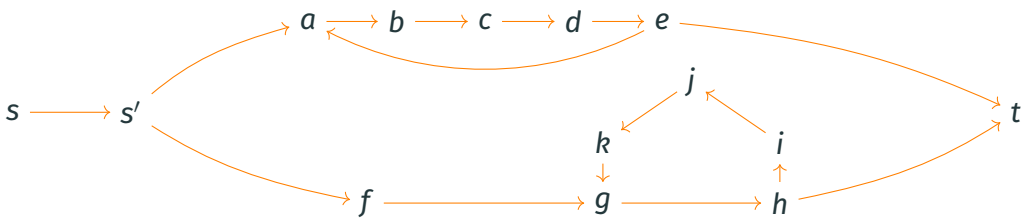

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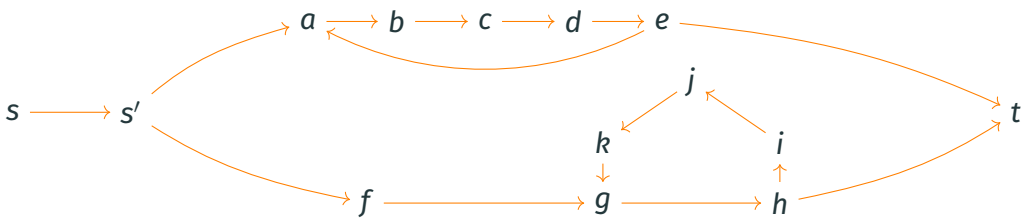

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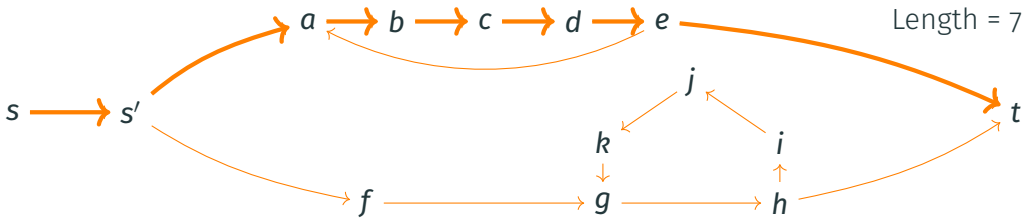

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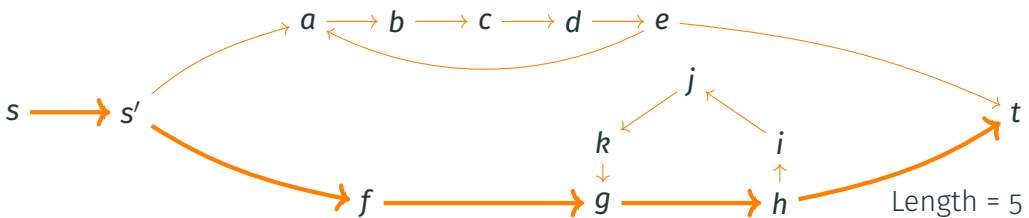

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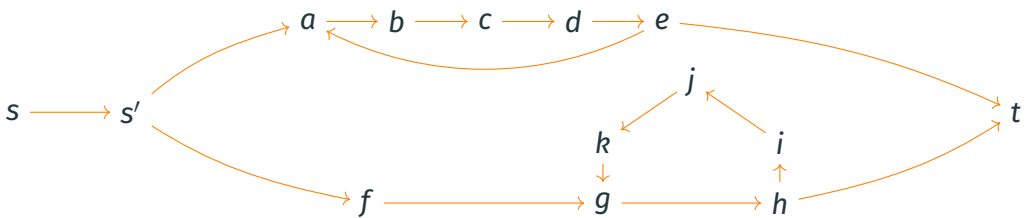

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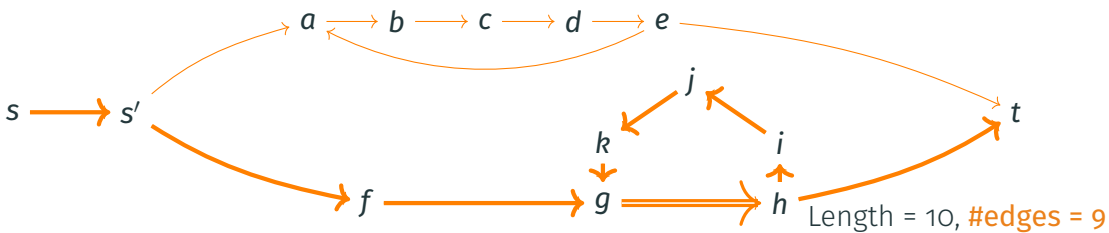

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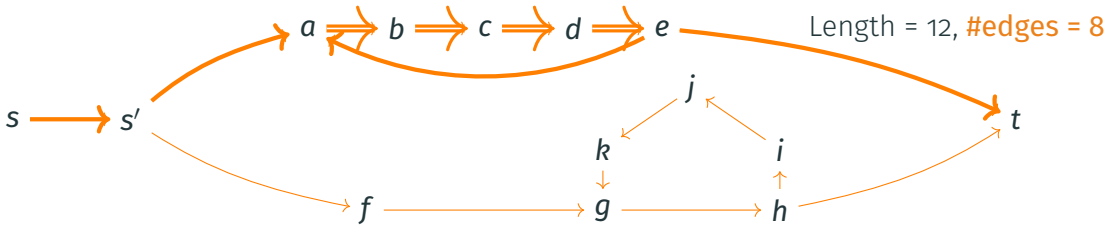

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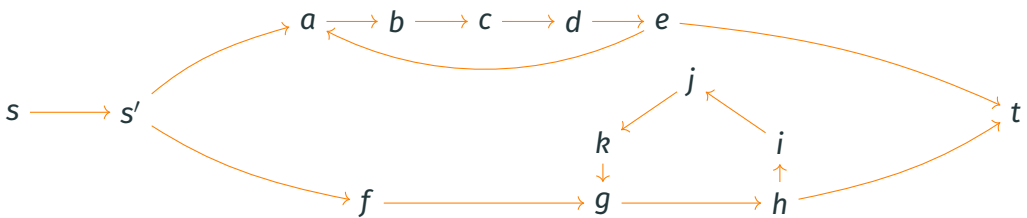

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- Is this problem in PTIME?

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→ Case 1: ***w*** has no cycle (and length = #edges)

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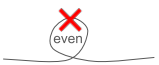

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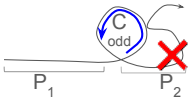

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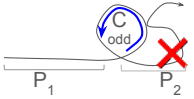

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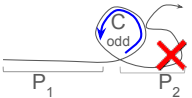

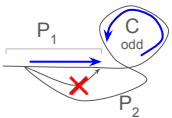

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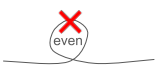

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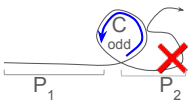

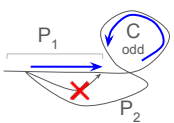

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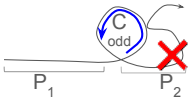

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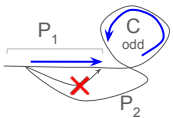

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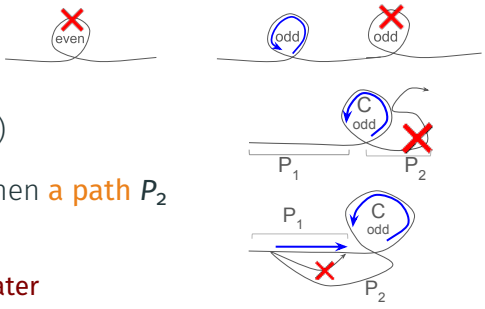

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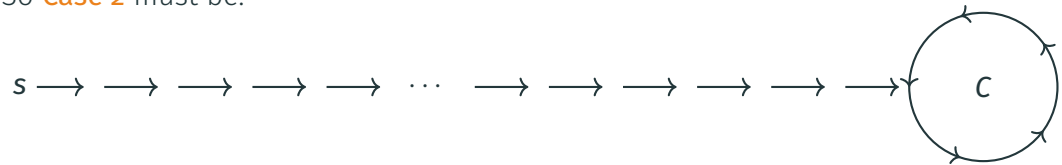

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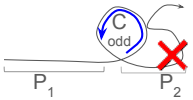

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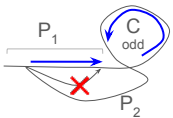

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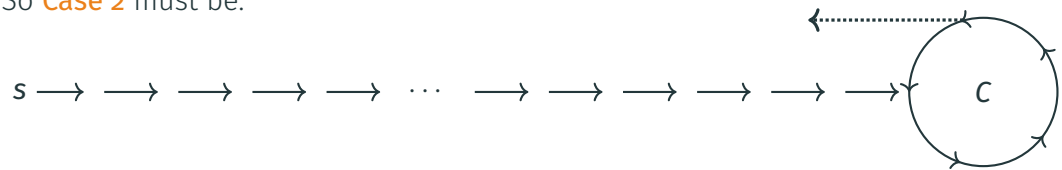

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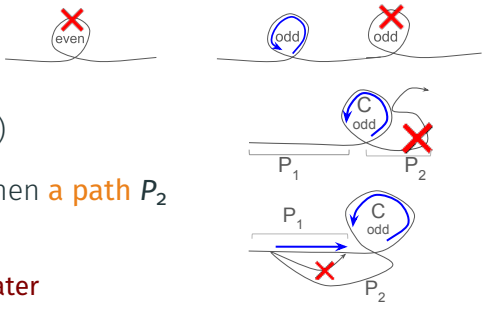

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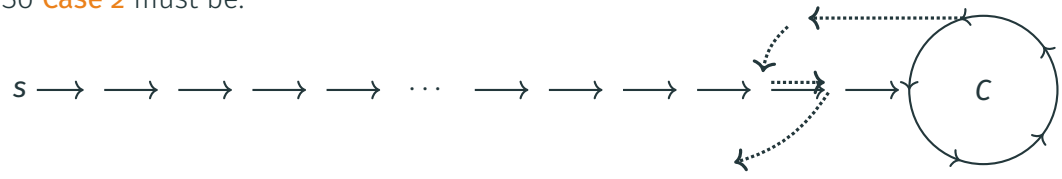

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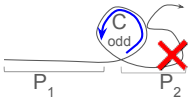

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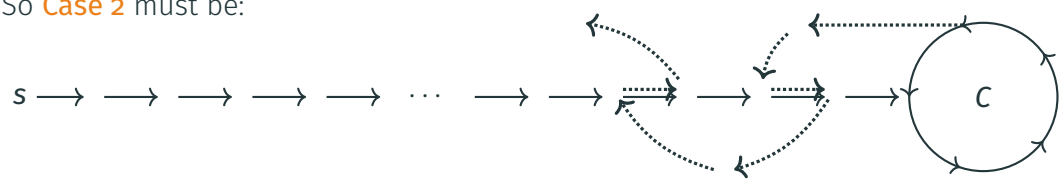

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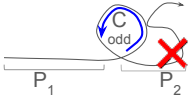

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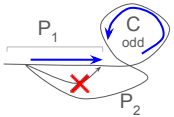

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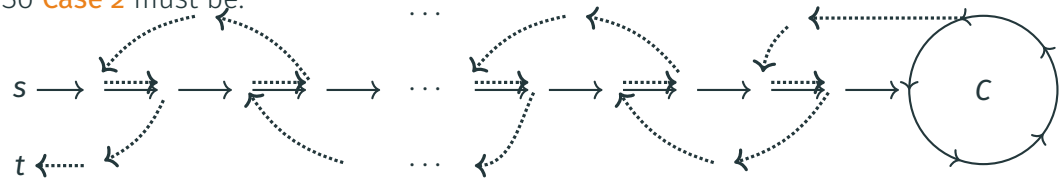

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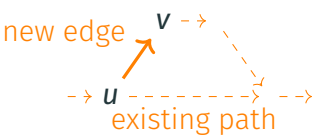

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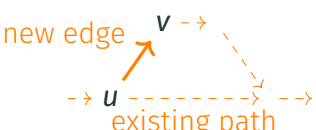

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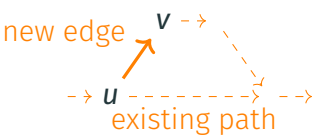

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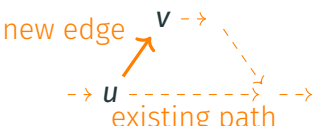

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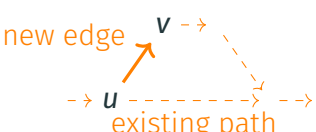

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Tractability for Modularity Constraints: Revisiting Parity (case $q = 2$)

- Optimal walk w has at most $O(\log q)$ detours: here 1
- Cutwidth of graph spanned by w is $O(\#\text{detours of } w)$: here 3
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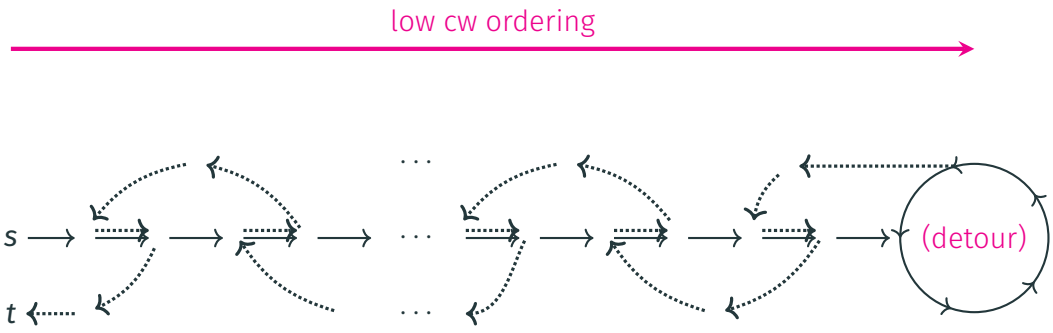

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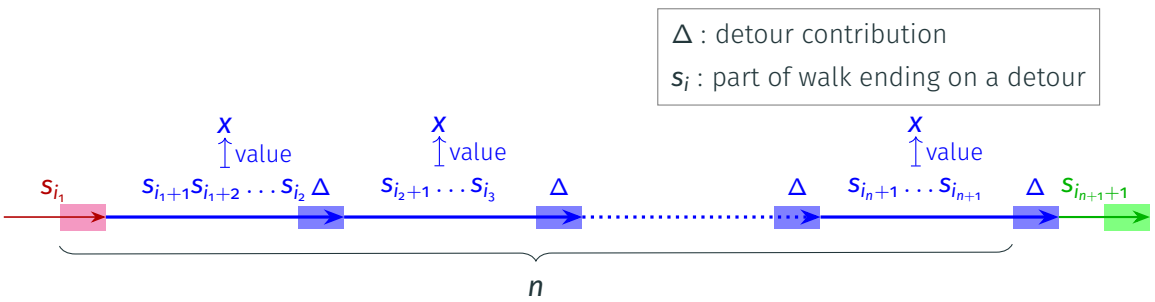
- After rewriting, the detours form a sequence of increasing subgroups of $\mathbb{Z}/q\mathbb{Z} \implies O(\log q)$ detours.

This proof is in fact valid on any finite commutative group!

What about other groups?

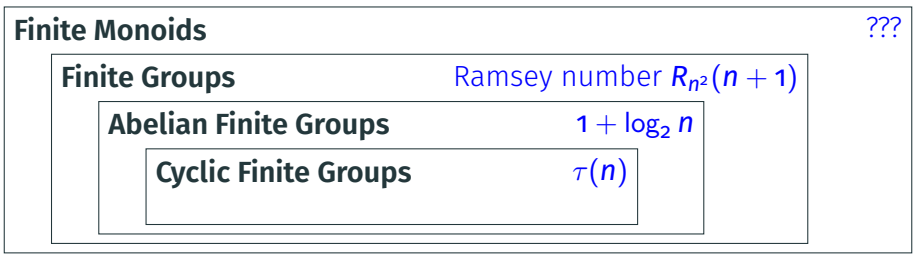
Non-Commutative Finite Groups

For finite groups of order n, Ramsey's Theorem guarantees n repetitions of (subwalk value, detour optional contribution) when #detours is sufficiently large



Bounds on Number of Detours, in a Nutshell

Upper bound on #detours (and cutwidth) in a smallest witness solution for monoids of order n:



$\tau(n)$ is the number of positive divisors of n (OEIS A000005)

Beyond a Single Walk (Work in Progress)

So far, our "few detours $\implies$ low cutwidth" argument is for a single walk so it does not capture Directed Steiner Network

More general problems:

- Directed Steiner Network with Modularity: $(a^{q_1})^* a^{r_1} b_1 (a^{q_2})^* a^{r_2} b_2 \cdots (a^{q_k})^* a^{r_k}$

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- Multilevel DSN: $a_1{}^* b_1 (a_1 + a_2)^* b_2 \cdots (a_1 + \cdots + a_n)^*$

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- Directed Steiner Network with Modularity: $(a^{q_1})^* a^{r_1} b_1 (a^{q_2})^* a^{r_2} b_2 \cdots (a^{q_k})^* a^{r_k}$
- Multilevel DSN: $a_1{}^* b_1 (a_1 + a_2)^* b_2 \cdots (a_1 + \cdots + a_n)^*$
- Multilevel DSN with Modularity...

Hardness results

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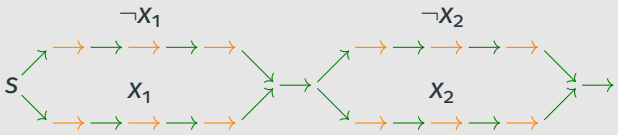

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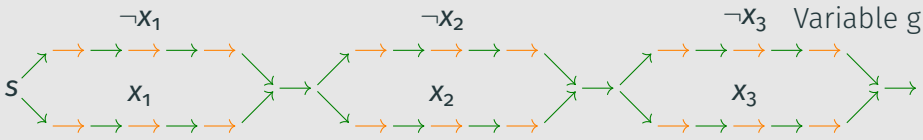

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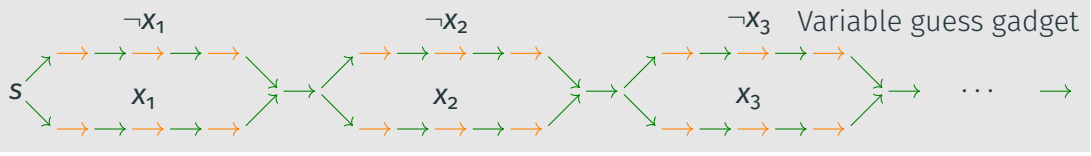

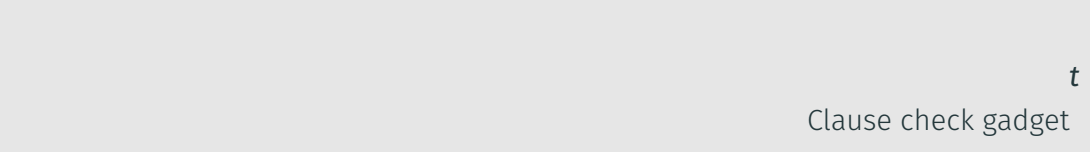

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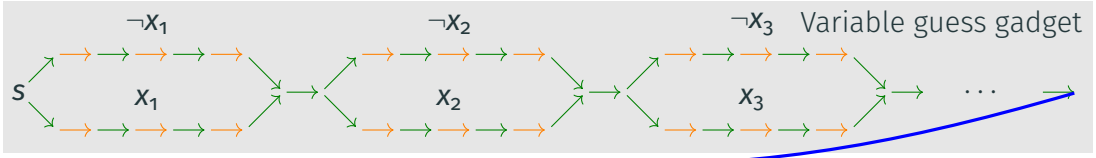

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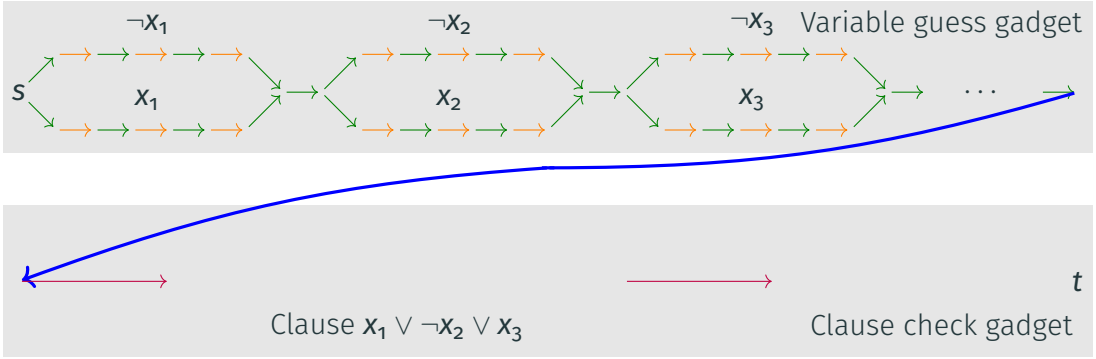

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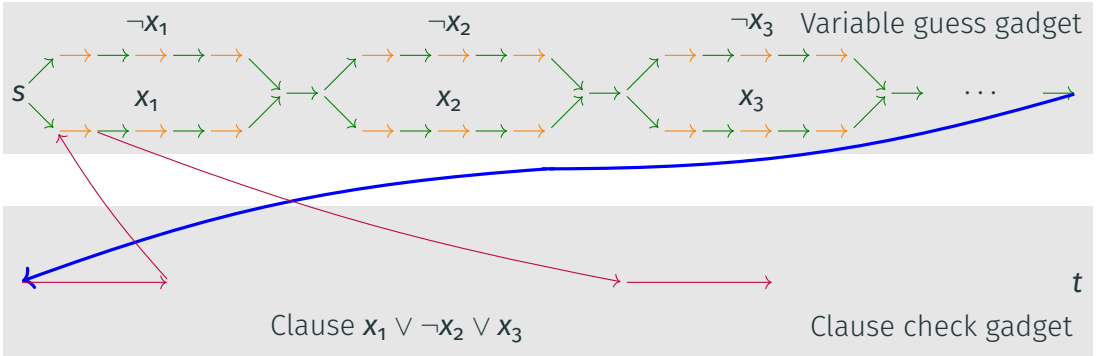

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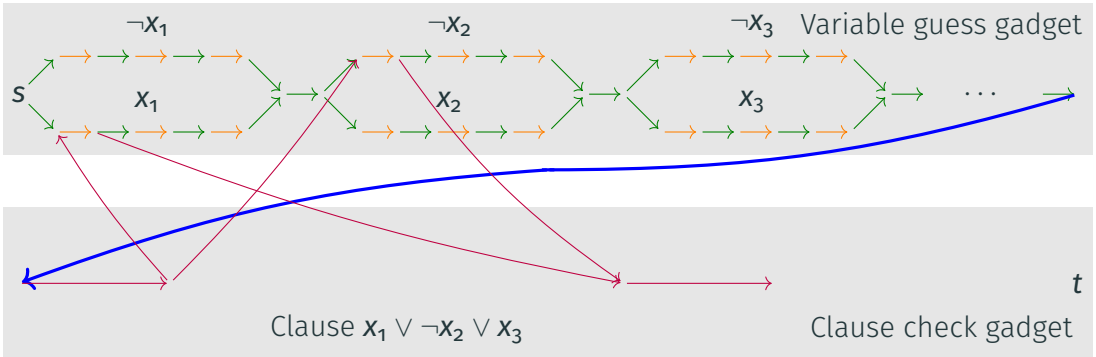

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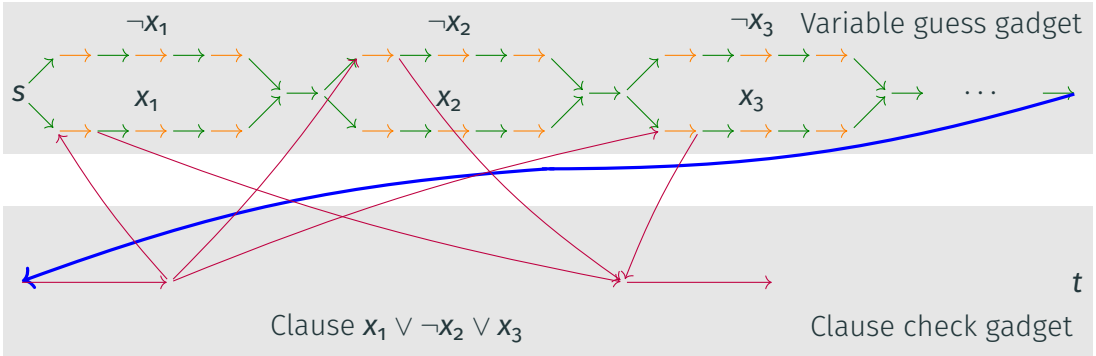

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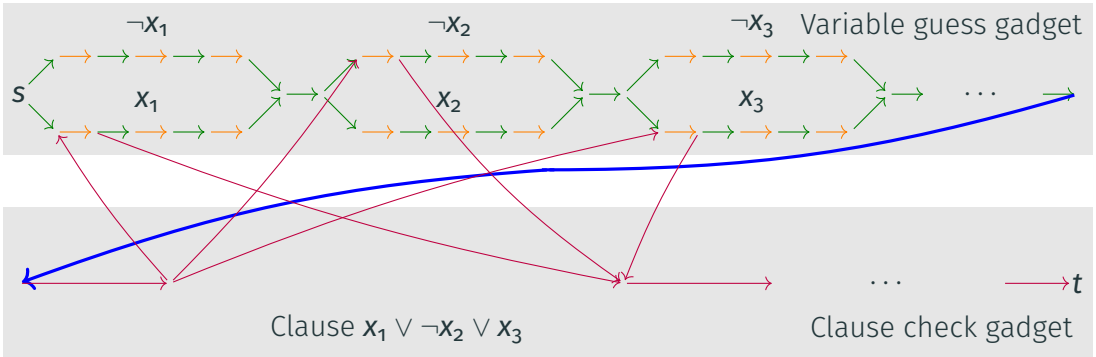

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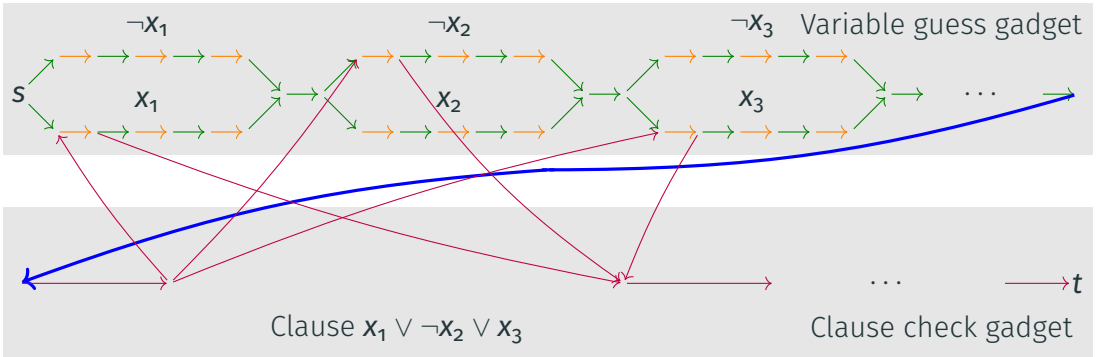

Minimizing #distinct edges "forces" a-edges in clause checks to be revisits
(All a-edges in clause-check are revisits $\Leftrightarrow$ 3-SAT instance satisfiable)

Open Question 1: Dichotomy on Smallest Witness for RPQs?

Conjecture (with Benoît Groz)

For every regular language L*, then* SW_L *is either in PTIME or NP-complete*

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- Tractability for the "multilevel" case: $a_1^* \, b_1 \, (a_1 + a_2)^* \, b_2 \cdots (a_1 + \cdots + a_n)^*$, possibly with modular/group constraints
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Possible problem variants:

- Size of witness = #edges (this talk) or #vertices
- Allowing edges labeled by ϵ, or by words of Σ^*...
- Weights on edges, other cost models, queries beyond RPQs, etc.

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Fix a monotone Boolean query Q, read as input a database D

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Thanks for your attention!

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Aparte 1: Side Result on Cutwidth

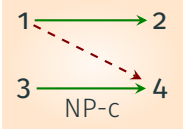
Theorem (based on [Feldmann and Marx, 2023])
Let $G = (V, E)$ be an undirected multigraph, P a partition of G, and G/P be the quotient multigraph of G under P. If the cutwidth of G/P is x and the cutwidth of every class of P is at most y, then the cutwidth of G is at most $1.5x + y$.

$\rightarrow$ constant 1.5 is optimal for a bound of this form (fixes [Feldmann and Marx, 2023]).

Aparte 2: Adding Negation

Constrained Reachability (CR(H)):
Fix: a directed "pattern" $H = (V_H, E_H^+, E_H^-)$.
Input: a directed graph $G = (V_G, E_G)$ and an (injective?) mapping $\phi : V_H \to V_G$.
Decide: is there a subgraph of G that contains a path from $\phi(s)$ to $\phi(t)$ for every $(s,t) \in E_H^+$ but no path from $\phi(u)$ to $\phi(v)$ for every $(u,v) \in E_H^-$?

1. If we fix the sizes $(|E_H^+|, |E_H^-|)$ only instead of the pattern H:
   - CR($1, \infty$) NP-c [Quesada and Brown, 2021]
   - CR($1, k$) in PTIME for constant k, even if we want a minimal witness.
   - CR($2, 1$) NP-c
2. Tractability by decomposing into reachability queries.
3. Hardness via reduction from 2 disjoint paths.



Is it related to known results? Can we find a dichotomy?

Connections to CSP/VCSP?

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- For smallest witness, unclear

SW_L: "For fixed RHS structure H_L, given a LHS structure G, find a minimal substructure of G that does not have a homomorphism to H_L"