Analogical Transfer: a Form of Similarity-Based Inference?

Fadi Badra
LIMICS, Paris, France

DIG seminar, March 5, 2018

Agenda

- Analogical Transfer : definition, examples
- Analogy and the Qualitative Measurement of Similarity
- Analogical Transfer : a Similarity-Based Inference ?
- Some more ideas I would like to share

Analogy

[Gust et al., 08] Analogy is a cognitive process in which a structural pattern identified in a source conceptualization is transferred to a target domain (possibly the same) in order to learn a target conceptualization.

3 main steps : Retrieval, Mapping, **Transfer**

[Holyoak and Thagard, 97] Analogy is guided by constraints on <u>similarity</u>, <u>structure</u> (*isomorphism* between some elements of source and target), <u>purpose</u> (the reasoner's goal)

+ <u>consistency</u> (solution viable in the reasoner's model of the world)

+ <u>simplicity</u> [Cornuéjols, 96]

Analogical Transfer

Assumption that if two situations are alike
in some respect, they may be alike in others

- Logical characterization [Davis and Russell, 87]

$$\frac{P(s) \quad P(t) \quad Q(s)}{Q(t)} \quad \text{(AJ)}$$

- What sufficient condition might justify "Analogical jump" (AJ) ?
  - weaker than generalization rule $\forall x, P(x) \Rightarrow Q(x)$
  - but stronger than single instance induction
  - depends on the amount of similarity between sources and targets
  - functional dependencies are good candidates

Let us take a few examples…

Example 1 : John's Car

$$\frac{P(s) \quad P(t) \quad Q(s)}{Q(t)}$$

s = Bob's car

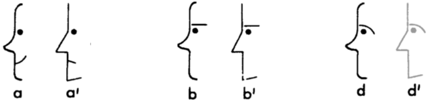
t = John's car

P="being a 1982 Mustang GLX V6 hatchbacks"

Q="having a price of 3 500 \$"

One can make the hypothesis that the price of t is $\approx$ 3 500 \$.

Example 2 : Tversky's Faces



$$\frac{R_1(\mathtt{a},\mathtt{a}') \quad R_2(\mathtt{a},\mathtt{a}') \quad R_1(\mathtt{d},\mathtt{d}')}{R_2(\mathtt{d},\mathtt{d}')}$$

R_1=" differs in profile shape from round to sharp "

R_2=" has the same eyebrow as "

Assuming that $\mathtt{d}'$ has a sharp profile (i.e., $R_1(\mathtt{d},\mathtt{d}')$ holds), one can make the hypothesis that $\mathtt{d}'$ has the same eyebrow as $\mathtt{d}$ (curved).

Example 3 : Analogical Classification

For $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{B}^n$,

$$\frac{\mathbf{a} : \mathbf{b} :: \mathbf{c} : \mathbf{d}}{cls(\mathbf{a}) : cls(\mathbf{b}) :: cls(\mathbf{c}) : cls(\mathbf{d})}$$

rewrites [Bounhas et al., 17]

$$\frac{P_k(\mathbf{a}, \mathbf{b}) \quad P_k(\mathbf{c}, \mathbf{d}) \quad Q_i(\mathbf{a}, \mathbf{b})}{Q_i(\mathbf{c}, \mathbf{d})}$$

with

- $P_k(\mathbf{a}, \mathbf{b}) = (\mathbf{a} - \mathbf{b} = \mathbf{k})$, with $\mathbf{k} \in \{-1, 0, 1\}^n$
- $Q_1(\mathbf{a}, \mathbf{b}) = (cls(\mathbf{a}) = cls(\mathbf{b}))$
- $Q_2(\mathbf{a}, \mathbf{b}) = (cls(\mathbf{a}) = 0 \wedge cls(\mathbf{b}) = 1)$
- $Q_3(\mathbf{a}, \mathbf{b}) = (cls(\mathbf{a}) = 1 \wedge cls(\mathbf{b}) = 0)$

Example 4 : the k-Nearest Neighbors

$$\frac{x \in \mathcal{N}(x) \quad cls(x) = c \quad x_0 \in \mathcal{N}(x)}{cls(x_0) = c}$$

- $\mathcal{N}(x)$ = neighborhood of x
- $cls(x)$ = class of x

Research Questions

How to formalize Analogical Transfer ?

- What should be transferred, and when ?
- Transfer may require adaptation (not only copy)
- Adaptation relies on comparison, an assessement of differences between source and target
- Adaptation knowledge include qualitative proportionalities ("*All things equal, the more an apartment has rooms, the more expensive it is*") or transformation rules ("*chocolate can be replaced by cocoa, but then sugar should be added*")

What is the role of similarity in Analogical Transfer ?

- Different ways to measure similarity : geometric, feature-based, alignment-based, transformational [Goldstone, 13] + visual

A Qualitative Measurement of Similarity

Qualitative Similarity Relations

- $\mathcal{U}$: a finite, non-empty set, the Universe
- We work on **pairs** (a, b) (or simply *ab*) of the square product $\mathcal{U} \times \mathcal{U}$
- Ordinal similarity relations [Yao, 00]

Strict Inequality

$ab \succ cd$ "*a* and *b* are more similar than *c* and *d*"

Equality

$ab \sim cd \Leftrightarrow \neg(ab \succ cd) \wedge \neg(cd \succ ab)$

"*a* and *b* are as similar as *c* and *d*"

Non-Strict Inequality

$ab \succeq cd$ iff $ab \succ cd$ or $ab \sim cd$

Scaling

A variation υ is a scale on $\mathcal{U} \times \mathcal{U}$

- Scale = a map that preserves $\geq$, *i.e.*, $ab \geq cd \Leftrightarrow \upsilon(ab) \geq \upsilon(cd)$
- Variations are used to **discretize** $\mathcal{U} \times \mathcal{U}$

$$
\begin{aligned}
Id &: xy \mapsto xy \\
\mathbf{1}_{\mathcal{U}} &: xy \mapsto 1 \\
=&: xy \mapsto \begin{cases} 1 & \text{if } x = y \\ 0 & \text{if } x \neq y \end{cases} \\
\mathbf{1} &: xy \mapsto \begin{cases} 1 & \text{if } x = y = 1 \\ 0 & \text{otherwise} \end{cases}
\end{aligned}
$$

Scaling

A variation υ is a scale on $\mathcal{U} \times \mathcal{U}$

$$d_p : xy \mapsto \|y - x\|_p$$

X, Y : *sets*

$$AP : XY \mapsto (X \backslash Y\ ,\ Y \backslash X)$$

$$\Delta : XY \mapsto |X \backslash Y| + |Y \backslash X|$$

Scaling

$\upsilon_{\mathbf{o}}^{\varphi}(ab) = \mathbf{o}(\varphi(a), \varphi(b))$ variation between values taken by φ

- first, apply a feature $\varphi : \mathcal{U} \to \mathcal{X}$ (e.g., age, gender)
- then, apply a scale $\mathbf{o} : \mathcal{X} \times \mathcal{X} \longrightarrow \mathcal{V}$ on the values of φ

$$\upsilon_{=}^{\texttt{eyebrow}}(ab) = \begin{cases} 1 & \text{if } \texttt{eyebrow}(a) = \texttt{eyebrow}(b) \\ 0 & \text{if } \texttt{eyebrow}(a) \neq \texttt{eyebrow}(b) \end{cases}$$

$$\mathbf{1}_P(ab) = \upsilon_{\mathbf{1}}^{P}(ab) = \begin{cases} 1 & \text{if both } P(a) \text{ and } P(b) \text{ hold} \\ 0 & \text{otherwise} \end{cases}$$

$$\upsilon_{d_1}^{\texttt{age}}(ab) = |\texttt{age}(b) - \texttt{age}(a)|$$

Scaling

$\upsilon_{\mathbf{o}}^{\varphi}(ab) = \mathbf{o}(\varphi(a), \varphi(b))$ variation between values taken by φ

Difference between two vectors

$$\mathbf{v}(a) = (\varphi_1(a), \varphi_2(a), \ldots, \varphi_n(a)) \in \mathbb{R}^n \text{ for } a \in \mathcal{U}$$

$$\overrightarrow{ab} = \upsilon_{-}^{\mathbf{v}}(ab) = \mathbf{v}(b) - \mathbf{v}(a)$$

Ranking difference (from a given object a)

$$\varphi(p) = size(\downarrow ap) \text{ with } \downarrow ap = \{ab \in \{a\} \times \mathcal{U} \mid ap \geq ab\})$$

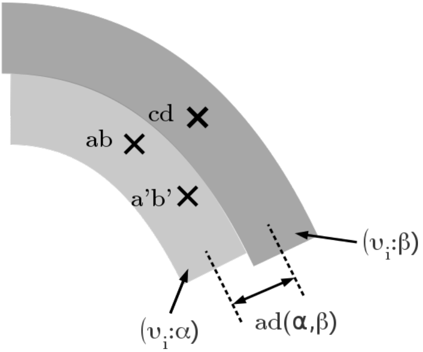
$$\upsilon_{d_1}^{a\rightarrow}(bc) = |size(\downarrow ac) - size(\downarrow ab)|$$

Nearest neighbor (b=a)

$$\upsilon_{d_1}^{a\rightarrow}(ac) = |size(\downarrow ac) - 1| = k \text{ iff } c \text{ is the k-NN of } a$$

Relation to Analogy ?

Intuition



Analogical Equalities

$\upsilon_i(ab) \sim \upsilon_i(a'b')$

being in a same contour line

Analogical Inequalities

$\upsilon_i(ab) \leq \upsilon_i(cd)$

defines contour scales

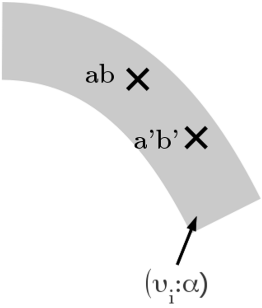
Analogical Dissimilarity

$\mathtt{ad}(ab, cd) = \mathtt{ad}(\alpha, \beta)$

distance between contour lines

Analogical Equalities

$\upsilon(ab) \sim \upsilon(a'b') \rightarrow$ "Contour lines" = sets of pairs equivalent for υ



Analogical Equalities

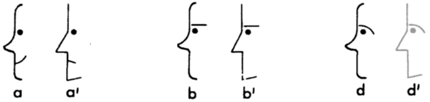
$\upsilon(ab) \sim \upsilon(a'b')$ → "Contour lines" = sets of pairs equivalent for υ

$\upsilon_{=}^{\mathtt{eyebrow}}(\mathtt{aa}') = \upsilon_{=}^{\mathtt{eyebrow}}(\mathtt{bb}') = 1$

`aa'` *and* `bb'` *both share a commonality : having the same eyebrow.*

$\upsilon_{ld}^{\mathtt{profile}}(\mathtt{aa}') = \upsilon_{ld}^{\mathtt{profile}}(\mathtt{bb}') = (\mathtt{round}, \mathtt{sharp})$

`aa'` *and* `bb'` *both share a difference : from round to sharp profile.*



Analogical Equalities

$\upsilon(ab) \sim \upsilon(a'b') \rightarrow$ "Contour lines" = sets of pairs equivalent for υ

X, Y : *sets*

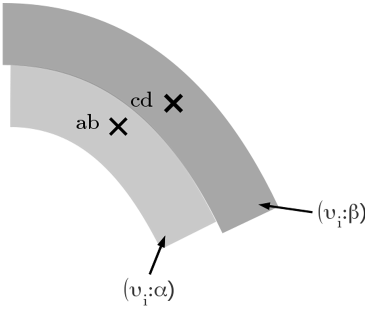
$$AP(XY) = (X \backslash Y\,,\ Y \backslash X)$$

$$AP(XY) = AP(ZT) \Leftrightarrow_{def} X : Y :: Z : T$$

X is to Y what Z is to T.

Analogical Inequalities

$\upsilon(ab) \leq \upsilon(cd) \rightarrow$ "Contour scales" = ordering on values of υ



Analogical Inequalities

$\upsilon(ab) \leq \upsilon(cd) \rightarrow$ "Contour scales" = ordering on values of υ

$$1 \geq 0 \Leftrightarrow (\upsilon_{=}^{\texttt{eyebrow}} : 1) \succeq (\upsilon_{=}^{\texttt{eyebrow}} : 0)$$

Two faces having the same eyebrow are more similar than two faces having different eyebrows.

$$n \leq m \Leftrightarrow (\upsilon_{d_1}^{\texttt{age}} : n) \succeq (\upsilon_{d_1}^{\texttt{age}} : m)$$

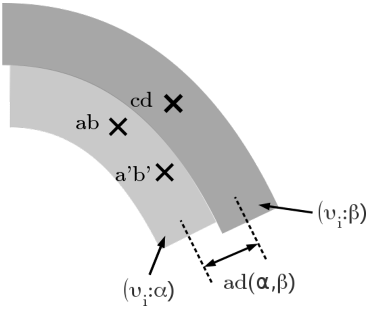
The lower the age difference, the more similar.

$$\ell \subseteq k \Leftrightarrow (AP : \ell) \succeq (AP : k)$$

All things equal, the less properties are lost or gained when going from a set X to a set Y, the more similar X and Y are.

Analogical Dissimilarity

$\texttt{ad}(ab, cd) \rightarrow$ measures the distance between two contour lines



Analogical Dissimilarity

$\mathtt{ad}(ab, cd) \rightarrow$ measures the distance between two contour lines

Dissimilarity between Nearest Neighbors

$$\upsilon_{d_1}^{a\rightarrow}(bc) = |size(\downarrow ac) - size(\downarrow ab)|$$

$$\mathtt{ad}(ab, cd) = \upsilon_{d_1}^{\upsilon_{d_1}^{a\rightarrow}}(ab, cd) \qquad \text{for } a, b, c, d \in \mathcal{U}$$

$$\textit{i.e.,} \qquad \mathtt{ad}(n, m) = |n - m| \qquad \text{for } n, m \in \mathbb{N}$$

AD is a ranking difference.

Analogical Dissimilarity

$\mathtt{ad}(ab, cd) \rightarrow$ measures the distance between two contour lines

Dissimilarity between Vectors

$$\mathbf{v}(a) = (\varphi_1(a), \varphi_2(a), \ldots, \varphi_n(a)) \in \mathbb{R}^n \text{ for } a \in \mathcal{U}$$

$$\overrightarrow{ab} = \upsilon^{\mathbf{v}}_{-}(ab) = \mathbf{v}(b) - \mathbf{v}(a)$$

$$\mathtt{ad}(ab, cd) = \upsilon^{\upsilon^{\mathbf{v}}_{-}}_{d_p}(ab, cd) = \|\overrightarrow{cd} - \overrightarrow{ab}\|_p$$

i.e., $$\mathtt{ad}(\overrightarrow{u}, \overrightarrow{v}) = \|\overrightarrow{v} - \overrightarrow{u}\|_p$$

Consistent with AD defined in [Miclet et al., 2008].

Analogical Dissimilarity

$\mathtt{ad}(ab, cd) \rightarrow$ measures the distance between two contour lines

Dissimilarity between Analogical Proportions

υ_{AP}^{φ} with $AP(XY) = (X \backslash Y\,,\ Y \backslash X)$

$$\begin{aligned}\mathtt{ad}(ab, cd) &= \upsilon_{\Delta}^{\upsilon_{AP}^{\varphi}}(ab, cd) \qquad (\Delta : \text{symmetric difference})\\ &= |U \backslash W| + |W \backslash U| + |V \backslash Q| + |Q \backslash V|\end{aligned}$$

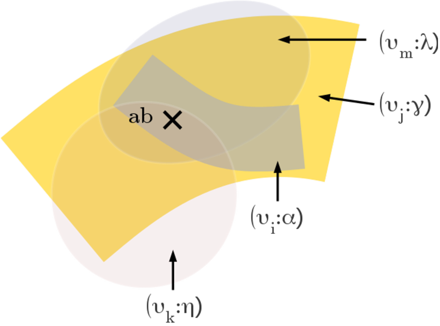
with $U = X \backslash Y,\ V = Y \backslash X,\ W = Z \backslash T,$ and $Q = T \backslash Z$

where $X = \varphi(a),\ Y = \varphi(b),\ Z = \varphi(c),$ and $T = \varphi(d)$

Consistent with AD defined in [Miclet et al., 2008].

Analogical Transfer

Intuition



Co-variation $\upsilon_i \overset{ab}{\rightharpoondown} \upsilon_j$

inclusion of contour lines

Analogical Transfer

$$\frac{\upsilon_i \overset{ab}{\rightharpoondown} \upsilon_j, \quad \mathtt{ad}(ab, cd) \leq k}{\upsilon_i \overset{cd}{\rightharpoondown} \upsilon_j}$$

similarity-based reasoning on contour lines

Co-Variations

$\upsilon_i \overset{ab}{\rightharpoondown} \upsilon_j \rightarrow$ inclusion between two contour lines

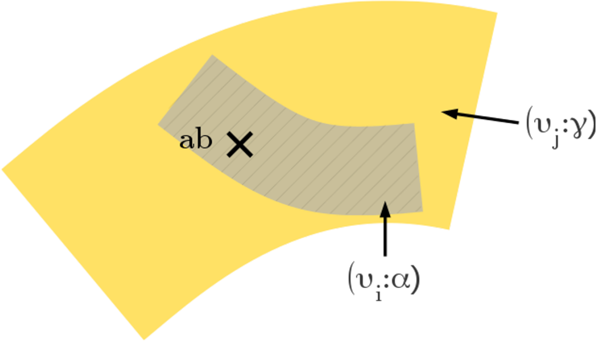
(υ_i co-varies with υ_j in ab)

Co-Variations

$\upsilon_i \overset{ab}{\curvearrowright} \upsilon_j \rightarrow$ inclusion between two contour lines

Bayesian semantics : $[ab]_{\upsilon_i} \subseteq [ab]_{\upsilon_j}$

inclusion is verified on the whole equivalence class of *ab*

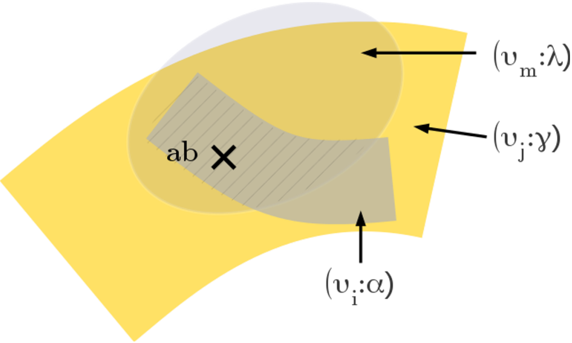


Co-Variations

$\upsilon_i \overset{ab}{\rightsquigarrow} \upsilon_j \rightarrow$ inclusion between two contour lines

Semi-Bayesian semantics : $\bigcap_{k \in \{i\} \cup D} [ab]_{\upsilon_k} \subseteq [ab]_{\upsilon_j}$

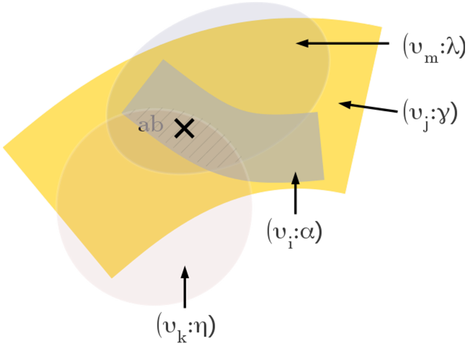
inclusion is verified around *ab*, but as long as we stay in the same contour line for some υ_k's



Co-Variations

$\upsilon_i \overset{ab}{\curvearrowright} \upsilon_j \rightarrow$ inclusion between two contour lines

Ceteris paribus semantics : $\bigcap_{k \neq j} [ab]_{\upsilon_k} \subseteq [ab]_{\upsilon_j}$
inclusion is verified *ceteris paribus* around *ab*, as long as we stay in the same contour line for *all other* υ_k's



Co-Variations

$\upsilon_i \overset{ab}{\rightharpoondown} \upsilon_j$ → inclusion between two contour lines

- Local implementation of the similarity principle :
  $\upsilon_i(ab) \sim \upsilon_i(cd) \Rightarrow \upsilon_j(ab) \sim \upsilon_j(cd)$
  "Similar problems have similar solutions"

  If $\sim_{\upsilon_i} = \sim_{\upsilon_j} = Id$:

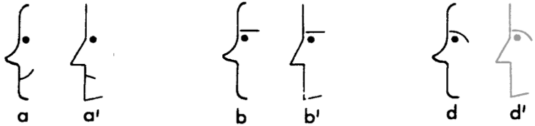
  - $(\upsilon_i : \ell) = \{ab \in \mathcal{U} \times \mathcal{U} \mid \upsilon_i(ab) = \ell\}$
  - $(\upsilon_i : \ell) \rightharpoondown (\upsilon_j : \gamma)$ is a functional dependency

Co-Variations

Example of Faces

$$(\upsilon_{\mathit{ld}}^{\texttt{profile}} : (\texttt{round}, \texttt{sharp})) \overset{\{\texttt{aa}',\texttt{bb}'\}}{\rightharpoonup} (\upsilon_{=}^{\texttt{eyebrow}} : 1)$$

When the profile changes from round to sharp, the eyebrow stays the same.



Co-Variations

Analogical Classification

$$\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{B}^n$$
$$\overrightarrow{ab} = \upsilon^{\mathbf{v}}_{-}(ab) = \mathbf{v}(b) - \mathbf{v}(a) \in \{-1, 0, 1\}^n$$
$$\upsilon^{cls}_{Id}(ab) = (cls(a), cls(b))$$

$$\upsilon^{\mathbf{v}}_{-} \overset{\mathbf{cd}}{\rightharpoonup} \upsilon^{cls}_{Id}$$

"A same difference between two (Boolean) vectors leads to a same difference of class."

Co-Variations

Co-monotony of partial orders

$(\upsilon_{\leq}^{\texttt{degree}} : 1) \overset{\texttt{bw}}{\rightharpoondown} (\upsilon_{\leq}^{\texttt{legal\_age}} : 1)$

The minimum legal age for alcohol consumption increases with the degree of alcohol.

k-Nearest Neighbors

$(\upsilon_{d_p}^{x_{\rightarrow}} : k) \overset{xx_0}{\rightharpoondown} (\upsilon_{=}^{cls} : 1)$

x_0 and its k-nearest neighbor share the same class.

Analogical Transfer

- Similarity principle taken as a local inference rule

$$\frac{\upsilon_i \overset{ab}{\rightharpoonup} \upsilon_j, \quad \mathtt{ad}(ab, cd) \leq k}{\upsilon_i \overset{cd}{\rightharpoonup} \upsilon_j}$$

  " If a contour line $[ab]_{\upsilon_i}$ is included in $[ab]_{\upsilon_j}$ at *ab*, then it should also be included at other points *cd* that are not too dissimilar"

- When υ_i's are identities :

$$\frac{(\upsilon_i : \ell_0) \overset{ab}{\rightharpoonup} (\upsilon_j : \gamma), \quad cd \in (\upsilon_i : \ell) \text{ such that } \mathtt{ad}(\ell_0, \ell) \leq k}{(\upsilon_i : \ell) \overset{cd}{\rightharpoonup} (\upsilon_j : \gamma)}$$

  " If a contour line $(\upsilon_i : \ell_0)$ is included in $(\upsilon_j : \gamma)$ at *ab*, then it should also be included at other points *cd* that are not too dissimilar"

Analogical Transfer

Analogical "Jump" (AJ)

$$\frac{P(s) \quad P(t) \quad Q(s)}{Q(t)}$$

rewrites

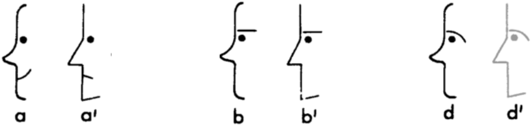
$$\frac{(\mathbf{1}_P : 1) \overset{ss}{\rightharpoondown} (\mathbf{1}_Q : 1), \quad st \in (\mathbf{1}_P : 1),\ \text{with } \mathtt{ad}(1,1) = 0}{(\mathbf{1}_P : 1) \overset{st}{\rightharpoondown} (\mathbf{1}_Q : 1)}$$

Analogical Transfer

Example of faces

$$\frac{(\upsilon_{ld}^{\texttt{profile}} : (\texttt{round}, \texttt{sharp})) \overset{\{\texttt{aa}', \texttt{bb}'\}}{\rightsquigarrow} (\upsilon_{=}^{\texttt{eyebrow}} : 1) \quad \texttt{dd}' \in (\upsilon_{ld}^{\texttt{profile}} : (\texttt{round}, \texttt{sharp}))}{(\upsilon_{ld}^{\texttt{profile}} : (\texttt{round}, \texttt{sharp})) \overset{\texttt{dd}'}{\rightsquigarrow} (\upsilon_{=}^{\texttt{eyebrow}} : 1)}$$

"If we know that d′ has a sharp profile, we can make the hypothesis that d′ has the same eyebrow as d (curved)."



Analogical Transfer

Analogical Classification

$$\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{B}^n$$

$$\overrightarrow{ab} = \upsilon^{\mathbf{v}}_{-}(ab) = \mathbf{v}(b) - \mathbf{v}(a) \in \{-1, 0, 1\}^n$$

$$\upsilon^{cls}_{Id}(ab) = (cls(a), cls(b))$$

$$\frac{(\upsilon^{\mathbf{v}}_{-} : \ell_0) \overset{\mathbf{ab}}{\rightarrow} (\upsilon^{cls}_{Id} : uv),\ \mathtt{ad}(ab, cd) = \|\overrightarrow{cd} - \overrightarrow{ab}\| = \|\overrightarrow{\ell} - \overrightarrow{\ell_0}\| \leq k}{(\upsilon^{\mathbf{v}}_{-} : \ell) \overset{\mathbf{cd}}{\rightarrow} (\upsilon^{cls}_{Id} : uv)}$$

*"The pair (cls(**a**),cls(**b**)) can be transferred from **ab** to **cd** whenever $\|\overrightarrow{cd} - \overrightarrow{ab}\|$ stays within a range ($\leq k$)."*

Analogical Transfer : Examples

A fortiori reasoning

$$\frac{(\upsilon_{\leq}^{\texttt{degree}} : 1) \overset{\texttt{bw}}{\rightsquigarrow} (\upsilon_{\leq}^{\texttt{legal\_age}} : 1), \quad \texttt{cb} \in (\upsilon_{\leq}^{\texttt{degree}} : 1)}{(\upsilon_{\leq}^{\texttt{degree}} : 1) \overset{\texttt{cb}}{\rightsquigarrow} (\upsilon_{\leq}^{\texttt{legal\_age}} : 1)}$$

"If we know that cider ($\mathtt{c}$) is less strong than beer ($\mathtt{b}$), then we can make the hypothesis that the minimum legal age for cider is less than the one for beer"

Analogical Transfer : Examples

k-Nearest Neighbors

$$\frac{(\upsilon_{d_p}^{x\rightarrow} : 0) \overset{xx}{\rightarrow} (\upsilon_{=}^{cls} : 1), xx_0 \in (\upsilon_{d_p}^{x\rightarrow} : n) \text{ with } \mathtt{ad}(0, n) \leq k}{(\upsilon_{d_p}^{x\rightarrow} : n) \overset{xx_0}{\rightarrow} (\upsilon_{=}^{cls} : 1)}$$

"The relation "belonging to the same class" is transferred from an element x to an element x_0 whenever x_0 is found to be in the neighborhood of x"

Analogical Transfer : a Form of Similarity-Based Inference ?

A Form of Similarity-Based Inference ?

- Analogical Transfer :

$$\frac{(\upsilon_i : \ell_0) \overset{ab}{\rightharpoonup} (\upsilon_j : \gamma), \quad cd \in (\upsilon_i : \ell) \text{ such that } \mathtt{ad}(\ell_0, \ell) \leq k}{(\upsilon_i : \ell) \overset{cd}{\rightharpoonup} (\upsilon_j : \gamma)}$$

- Similarity-Based Inference :

$$\frac{P \Rightarrow Q \quad P' \approx P \quad P'(t)}{Q(t)}$$

- $P = (\upsilon_i : \ell_0)$, $P' = (\upsilon_i : \ell) \rightarrow$ equivalence classes for υ_i
- $Q = (\upsilon_j : \gamma) \rightarrow$ equivalence class for υ_j
- $P' \approx P \rightarrow$ analogical dissimilarity within a range ($\mathtt{ad}(\ell_0, \ell) \leq k$)

Conclusion (take away)

- There is an intimate link between the qualitative measurement of similarity and formal methods of analogy :
  - Mapping = being in the same equivalence class (“contour line”) for some similarity relation υ
  - Analogical Inequalities = ordering on contour lines
  - Analogical Dissimilarity = distance between two contour lines
- Analogical Transfer can be seen as a similarity-based inference on some equivalence classes of ordinal similarity relations :
  - Co-variation = inclusion of two contour lines (= implementation of the similarity principle)
  - Analogical Transfer = similarity-based reasoning on a co-variation ($(\upsilon_i : \ell_0) \approx (\upsilon_i : \ell) \Leftrightarrow \mathtt{ad}(\ell_0, \ell) \leq k$)

Some Ideas

Idea 1 : Link with Simplicity Theory ?

- [Cornuéjols, 96] “Proximity between source and target is measured by the size of the minimal program that allows to derive target from source”
- Variations need not represent only feature differences, but could represent rewriting rules :
  $"\texttt{aabc}" \rightarrow "\texttt{aabd}" \quad \texttt{aa} \xrightarrow{\texttt{same}} \texttt{aa} \quad \texttt{b} \xrightarrow{\texttt{same}} \texttt{b} \quad \texttt{c} \xrightarrow{\texttt{successor}} \texttt{d}$
  $"\texttt{ijkk}" \rightarrow "\texttt{ijll}" \quad \texttt{i} \xrightarrow{\texttt{same}} \texttt{i} \quad \texttt{j} \xrightarrow{\texttt{same}} \texttt{j} \quad \texttt{kk} \xrightarrow{\texttt{successor}} \texttt{ll}$
  $\upsilon("\texttt{aabc}", "\texttt{aabd}") = \upsilon("\texttt{ijkk}", "\texttt{ijll}") = (\texttt{same}, \texttt{same}, \texttt{successor})$
  $"\texttt{aabc}" : "\texttt{aabd}" :: "\texttt{ijkk}" : "\texttt{ijll}"$
- Use minimum description length as an order on the values of υ ?
  $\ell = (\texttt{same}, \texttt{successor}) \geq \ell' = (\texttt{same}, \texttt{same}, \texttt{successor})$
  $\rightarrow$ best mapping is in contour line $(\upsilon : \ell) = (\upsilon : (\texttt{same}, \texttt{successor}))$

Idea 2 : Invoke Methods from Topology

Qualitative measurement of similarity can be the link between analogy and topological methods.

- Consider the ternary relation $T(a, b, c) \Leftrightarrow ab \geq ac$ ("b is at least as similar to a as c is")
- [Nehring, 97] A ternary relation $T(a, b, c)$ has a unique representation as a convex topology, and the latter is the collection of segments $s_{ac} = \{b \mid (a, b, c) \in T\}$
- Use this observation to invoke tools from :
  - topological data analysis → [Giavitto, 96]
  - visualization → [Lamy, 18]

Idea 3 : Formalize the Case-Based Inference

$$\begin{tikzcd}
\mathtt{a} \arrow[r, "(\upsilon_{\mathtt{pb}} : \ell)"] & s \\
\mathtt{b} \arrow[u] \arrow[r] & t \arrow[u, "\upsilon_{\mathtt{pb}}(t, s) = (\upsilon_{\mathtt{pb}} : \ell_0)"']
\end{tikzcd}$$

1. *Retrieval*. $tt \mapsto \mathcal{N}$: find a neighborhood $\mathcal{N}(tt)$ of tt that contains a pair st.
2. *Mapping*. $ts \mapsto (\upsilon_{\mathtt{pb}} : \ell_0)$: represent the problem differences from t to s as a contour line of a variation $\upsilon_{\mathtt{pb}}$.
3. *Transfer*. Take the set $S = \mathtt{argmin}_k\, ab \mapsto \mathtt{ad}(st, ab)$ that contains the k pairs that are most similar to the pair st with respect to $\upsilon_{\mathtt{pb}}$ and use the co-variations $(\upsilon_{\mathtt{pb}} : \ell) \rightharpoondown (\upsilon_{\mathtt{sol}} : \gamma)$ at ab to decide how to complete the solution

Thank you！