An Experimental Study of Treewidth of Real-World Graph Data

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LTCI Data Science Seminar, Paris, June 6th 2019

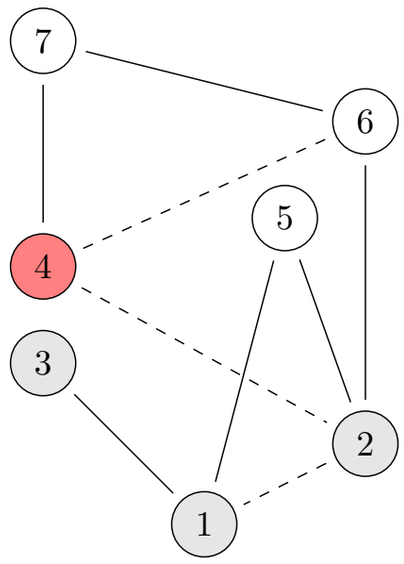
Why Treewidth?

Many computational tasks are hard (when considering data complexity) on graph-like structures, but are easy on tree structures

But just how tree-like is the data? And how useful is having this tree-likeness measure?

Application: Shortest Distances

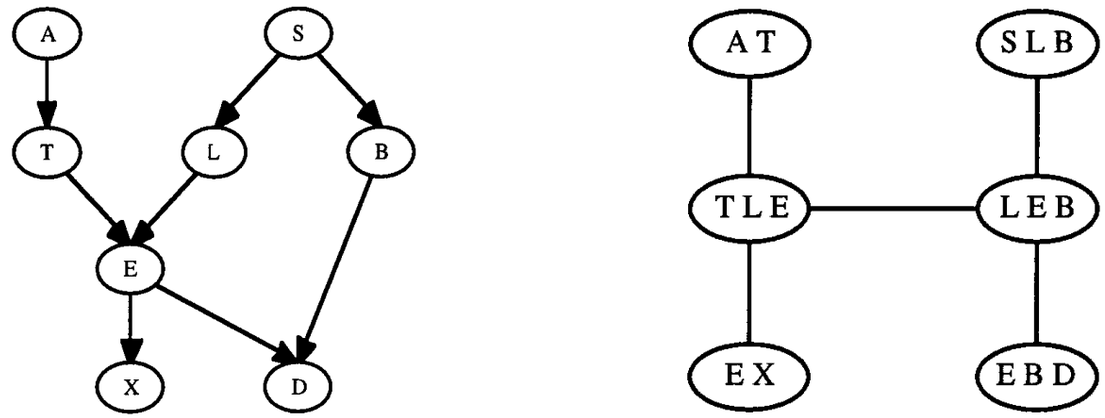
Computing all-pairs shortest distances on a graph has a complexity not lower than matrix multiplication (roughly cubic in the size of the graph)



By computing a triangulation of the graph, the complexity can be reduced to $\mathcal{O}(V^2 w_c)$— depending on the size of the largest clique in the triangulation [Planken et al., 2012]

Application: Junction Trees

Exact belief propagation in graphical models is known to be very costly in graphs ($\mathcal{O}(N^M)$ exponential in the number of states), and efficient on trees.



[Lepar and Shenoy, 2011]

Hugin's algorithm (exact belief propagation):

1. transform the graph into a tree (by clustering cliques) — junction tree

2. perform exact propagation inside the cliques and along the tree — exponential only in the size of the largest clique in the junction tree

Treewidth in Data

Treewidth: a measure on how tree-like data is [Robertson and Seymour, 1984]

For data that has bounded treewidth, one can obtain efficient algorithms in databases:

1. query evaluation of MSO queries is linear time over bounded-treewidth data structures [Courcelle, 1990], also applies to counting and enumeration

2. computing probabilities of MSO queries over TID probabilistic databases which are bounded-treewidth is linear time [Amarilli et al., 2015]

Treewidth: Intuition

Treewidth: a measure on how tree-like data is [Robertson and Seymour, 1984]

Can be obtained using a tree decomposition (or a hierarchy of separators)

The treewidh is the smallest obtainable size of a separator (minus one)

Treewidth: Intuition

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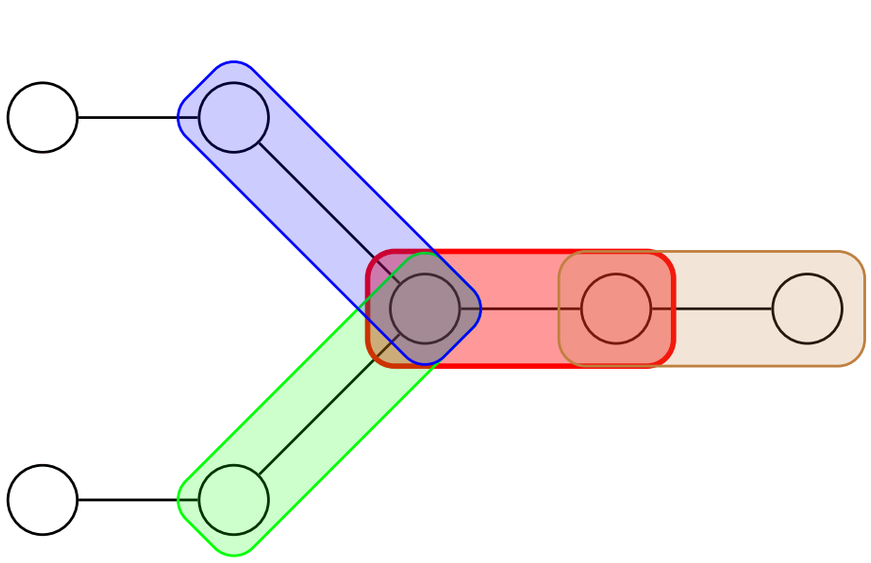
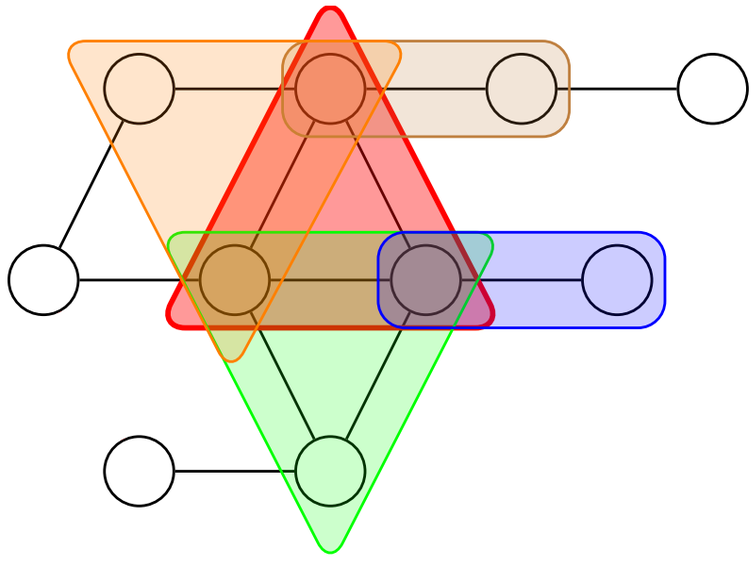




Treewidth: Intuition

The **treewidh** is the smallest obtainable size of a separator (minus one)

- trees have treewidth 1
- cycles have treewidth 2
- *k*-cliques have treewidth *k*-1

Treewidth: Formal Definition

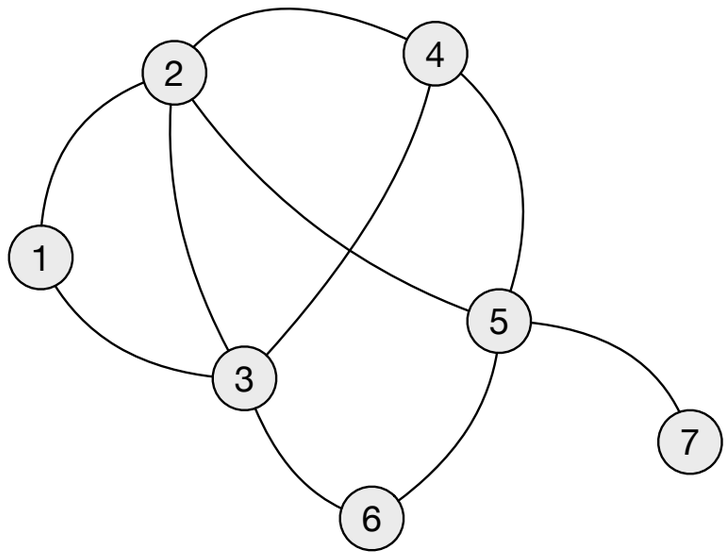
Definition

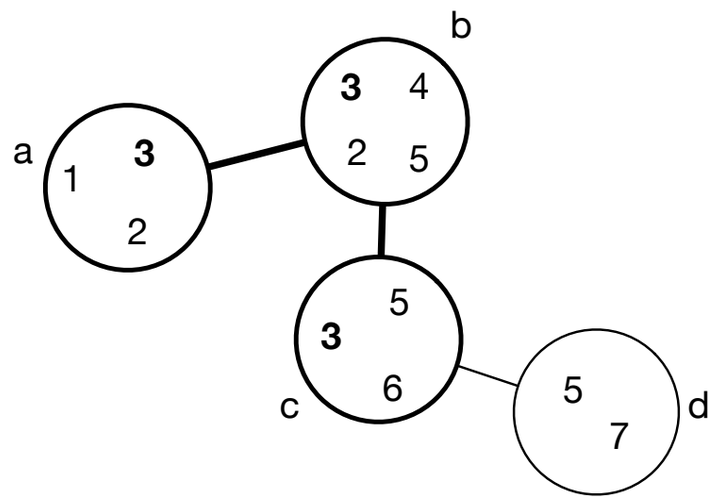
Given an *undirected graph* $G = (V, E)$, where V represents the set of vertices (or nodes) and $E \subseteq V \times V$ the set of edges, a *tree decomposition* is a pair (T, B) where $T = (I, F)$ is a tree and $B : I \to 2^V$ is a labeling of the nodes of T by subsets of V (called *bags*), with the following properties:

1. $\bigcup_{i \in I} B(i) = V$;
2. $\forall (u, v) \in E$, $\exists i \in I$ s.t. $\{u, v\} \subseteq B(i)$; and
3. $\forall v \in V$, $\{i \in I \mid v \in B(i)\}$ induces a subtree of T.

Treewidth: Formal Definition

Given a graph $G = (V, E)$ the *width* of a tree decomposition (T, B) is equal to $\max_{i \in I}(|B(i)| - 1)$. The *treewidth* of G, $w(G)$, is equal to the minimal width of all tree decompositions of G.





Treewidth

Readily usable for graphs, for relational (=database) data the Gaifman graph can be used

Only low treewidth makes things easy: even linear time algorithms hide a non-elementary dependency in treewidth

In some cases, treewidth is the only hope to have polynomial time algorithms! [Amarilli et al., 2016]

In Practice

If data has low treewidth, we have plenty of efficient algorithms (based on tree decompositions)

Question 1: Are real-world data low treewidth?

Computing Treewidth

Computing the treewidth exactly is hard [Arnborg et al., 1987]

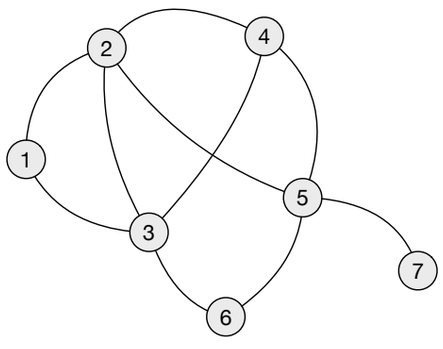
We can compute upper bounds [Bodlaender and Koster, 2010], which also give a tree decomposition

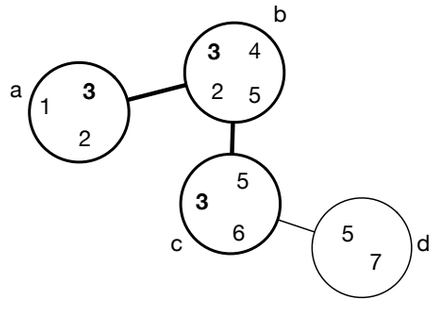
Also can obtain lower bounds efficiently [Bodlaender and Koster, 2011]

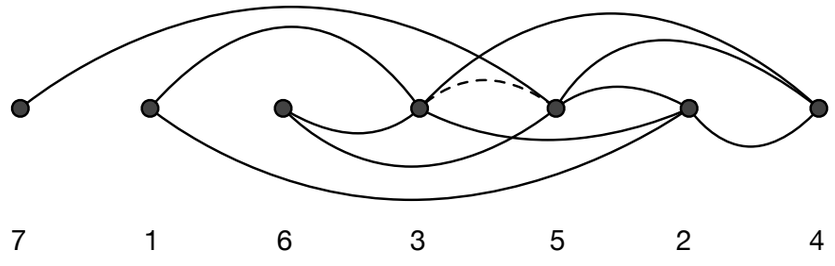
Upper Bound Algorithms

General algorithm, generating a tree decomposition:

1. generate an ordering of the nodes
2. for each node in the ordering, create a bag containing it and its neighbors
3. remove the node from the graph, create a clique between the neighbors
4. continue until no nodes are left



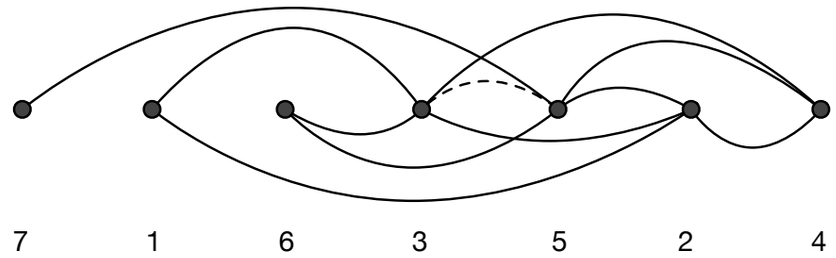




Upper Bound Algorithms

Details differ in how the ordering is generated:

- minimum degree first (MINDEGREE)
- minimum fill-in (MINFILLIN)
- combination thereof



Lower Bound Algorithms

Computing other proxy measures, which are lower bounds on treewidth:

1. second lowest degree in the graph (DELTA2D)

2. second lowest degree in a subgraph of the graph (MMD)

3. second lowest degree in a minor of the graph (MMD+)

Algorithms differ in how the subgraphs and minors are explored (usually in a greedy fashion)

Experimental Setup

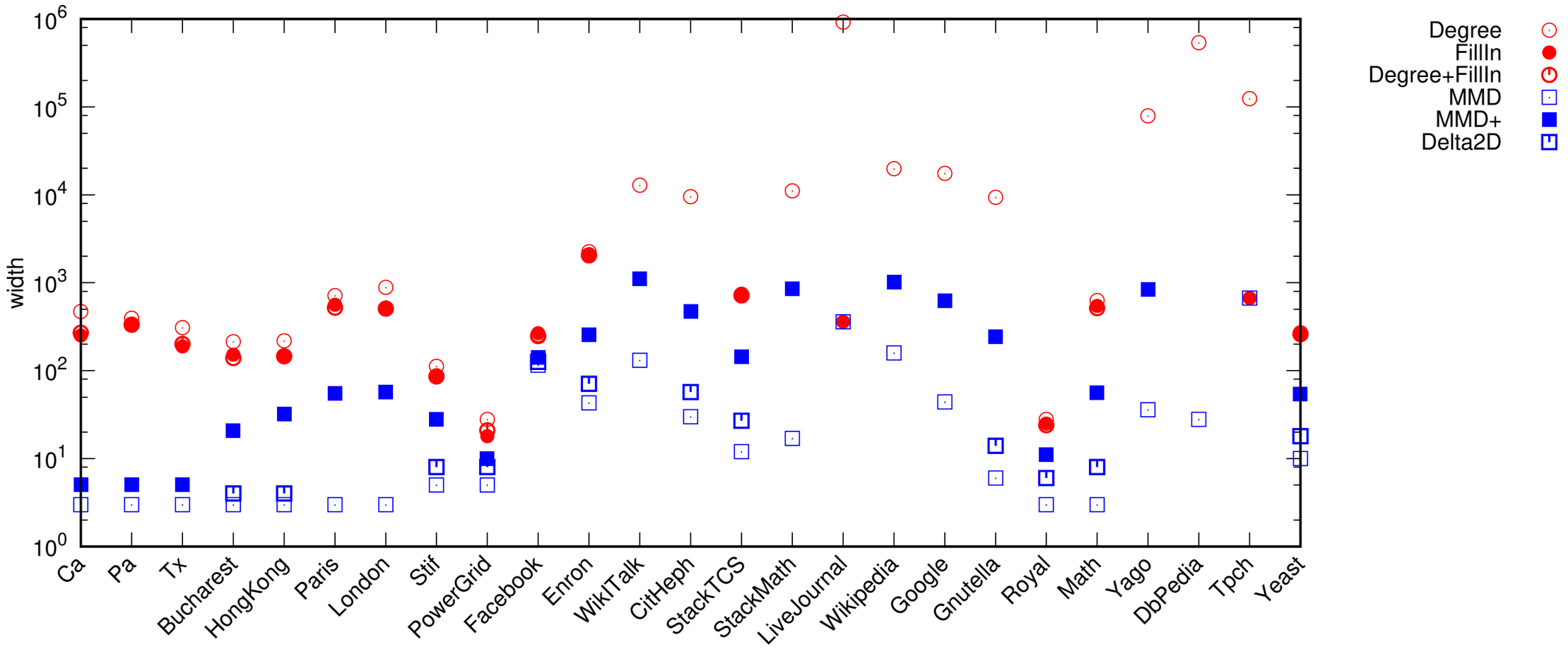
25 graph datasets from 8 different domains:
infrastructure, social networks, Web, communication, hierarchies, ontologies, relational databases, biology

Tests ran on machines having 32GB RAM, Intel Xeon 1.7GHz CPUs

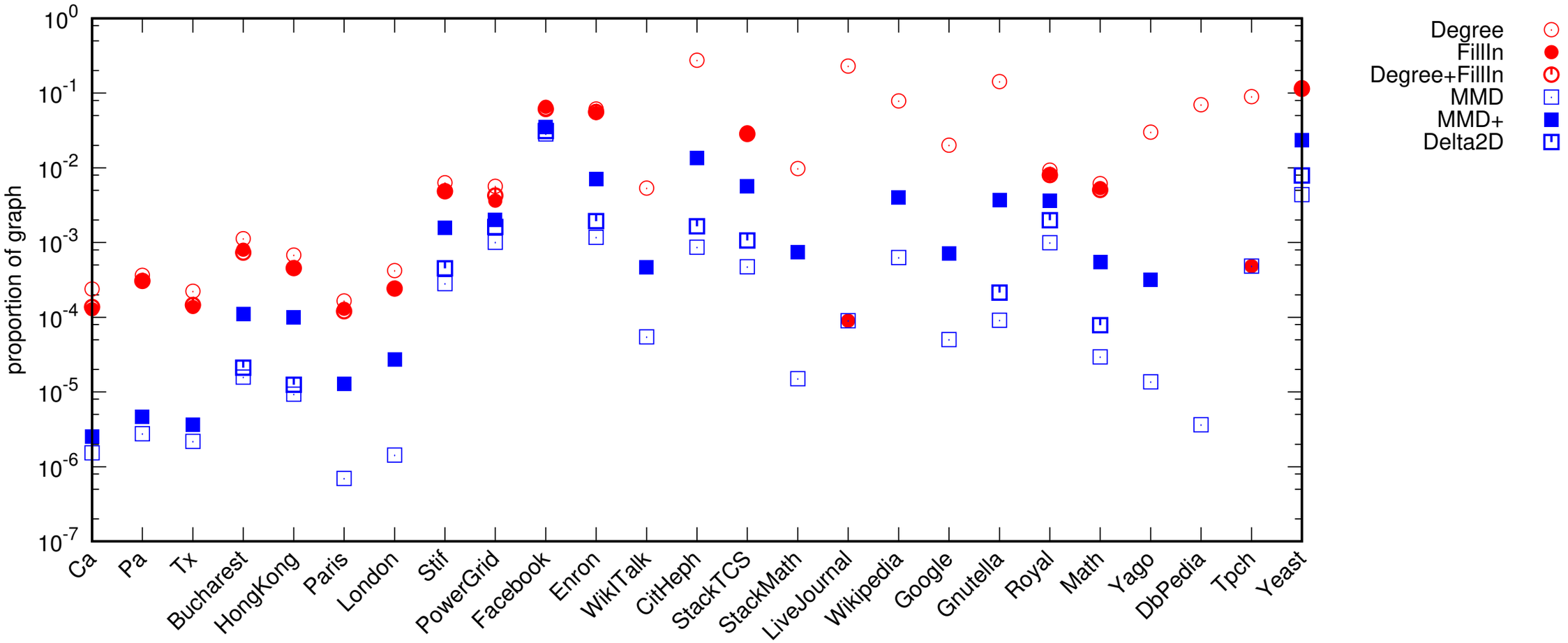
The maximal running time allowed was *two weeks*

Code and datasets available online
https://github.com/smaniu/treewidth

Treewidth: Absolute Values



Treewidth: Relative Values



Main Takeaways

Negative result: treewidth is never very low (<10)

However, for infrastructure networks, treewidth is relatively low (similar to the $\mathcal{O}(\sqrt[3]{n})$ bound conjectured in [Dibbelt et al, 2016])

Other graphs exhibit high treewidth values, outside practical usefulness: social networks, Web graphs, knowledge graphs

Is Treewidth Only of Theoretical Relevance?

The second question: can we still find practical use cases for treewidth (and tree decompositions)?

Partial Tree Decompositions

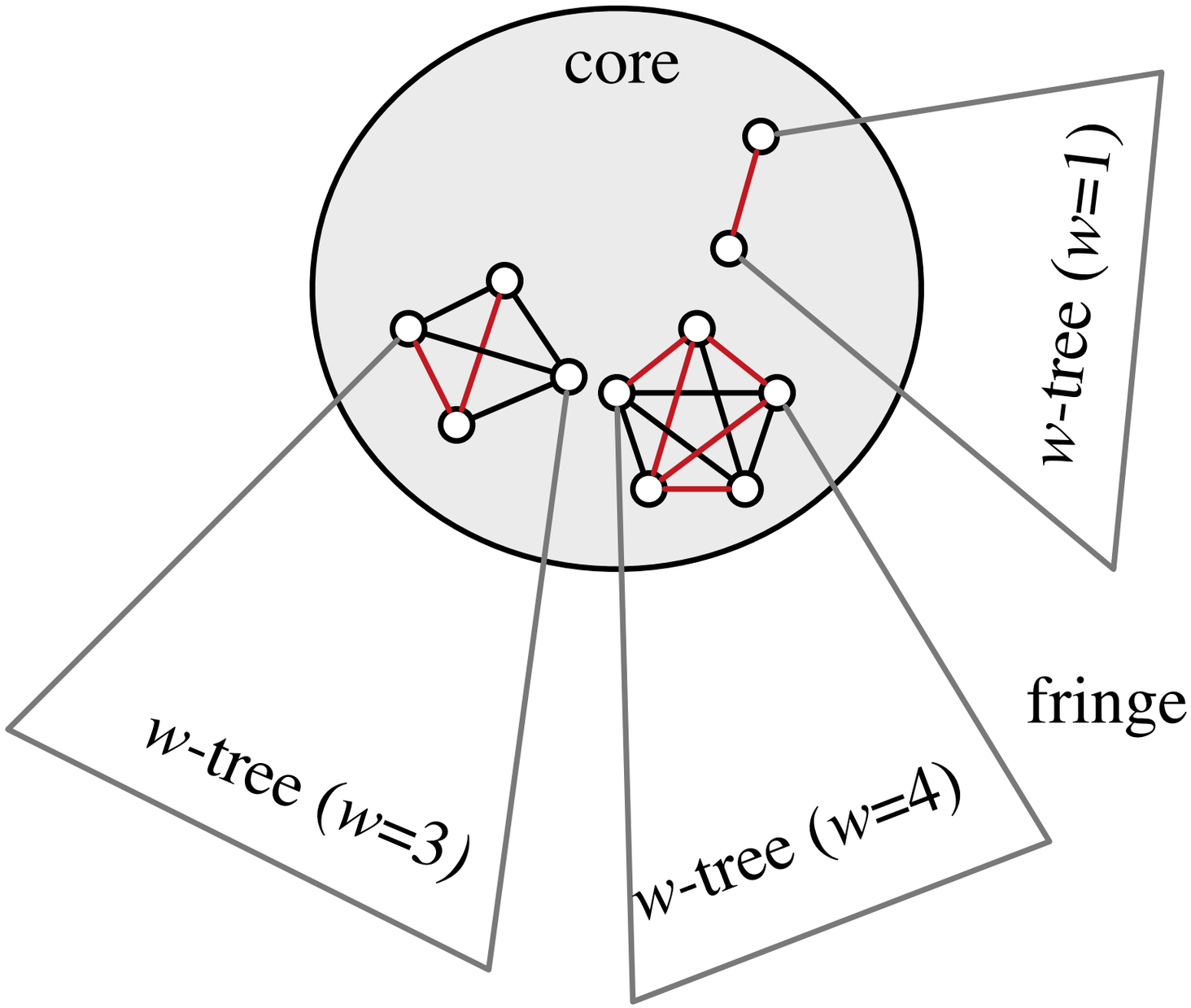
Instead of computing the full tree width estimation, what if we stop sometime during the decompositions process?
— Partial Tree Decomposition

Results in a hybrid tree-graph structure:

1. a low treewidth fringe, and

2. a (potentially) high-treewidth core graph

Partial Tree Decompositions



General Algorithm for Partial Tree Decompositions

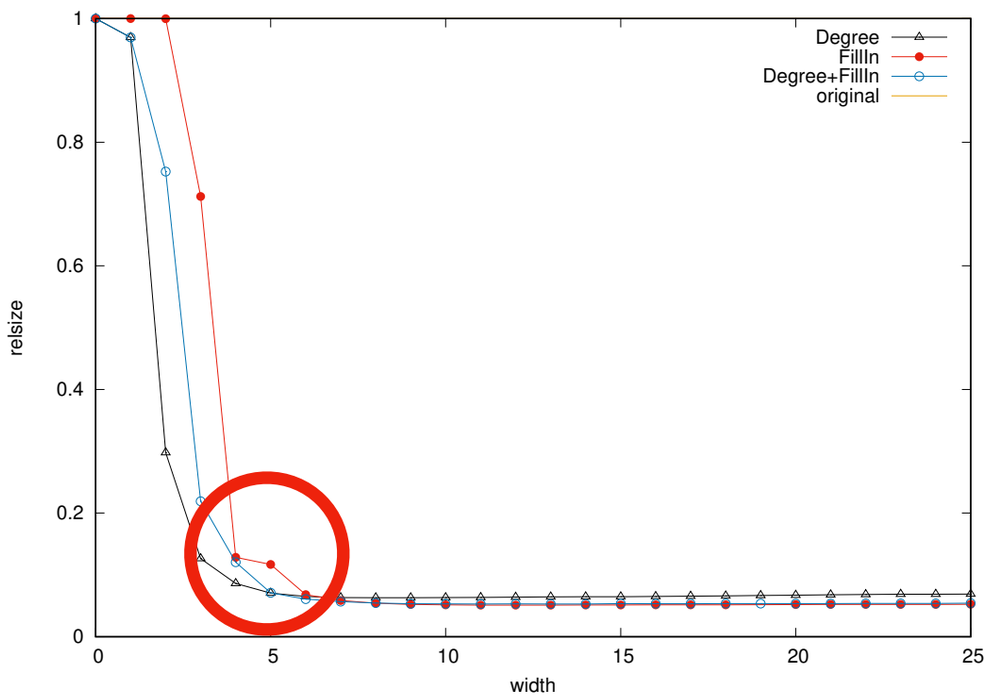
1. Isolate a part of low treewidth (by stopping the decomposition process early)

2. (Pre-)process the fringe efficiently (due to its low treewidth)

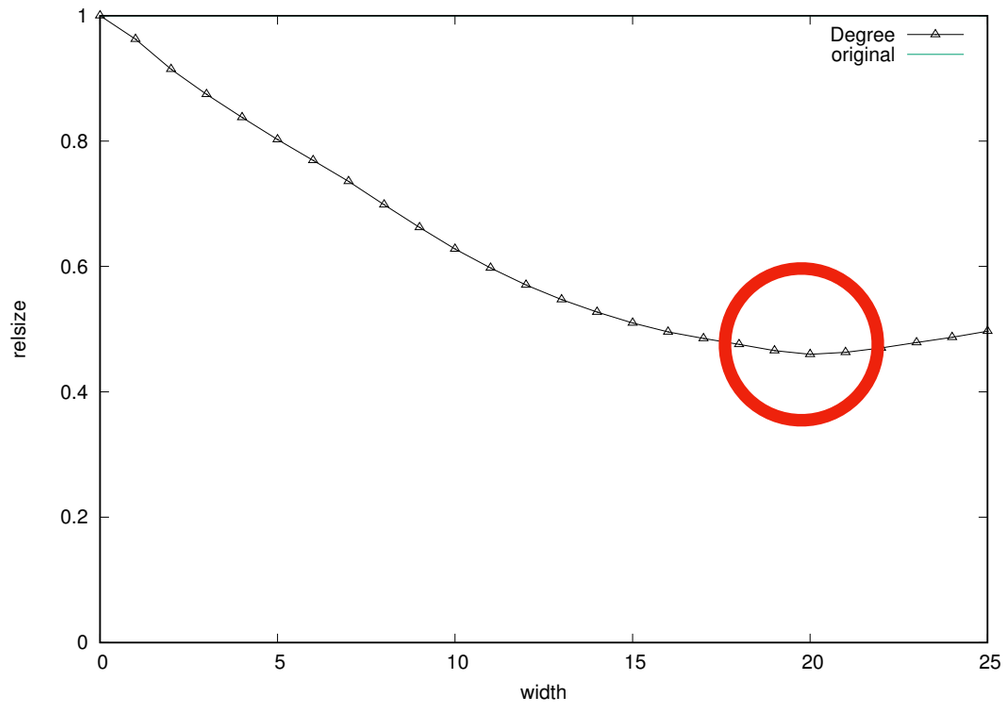
3. Process the core graph using other techniques (e.g., approximate algorithms)

4. Combine results

Partial Tree Decompositions: When to Stop?



Paris OpenStreetMaps
(4M road segments)

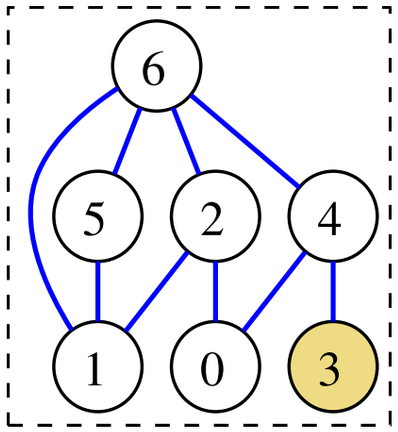


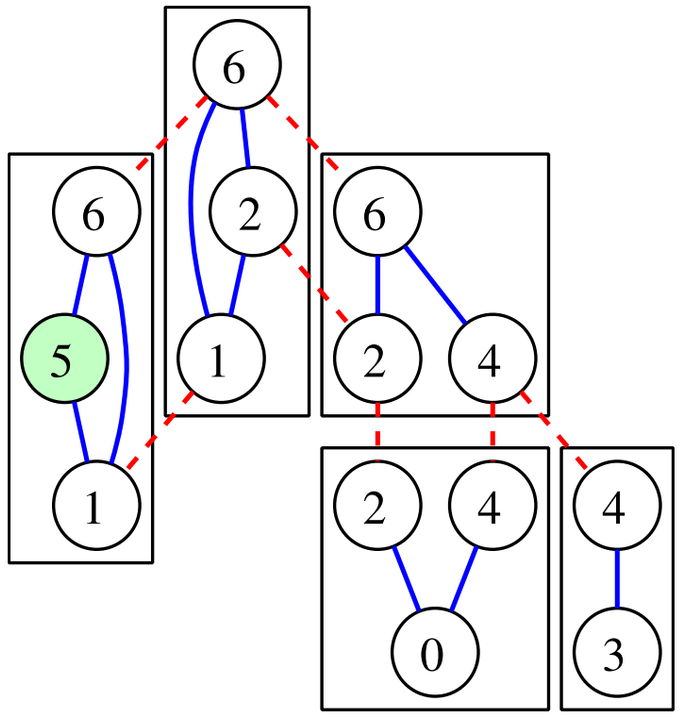
Google Web graph
(875k hyperlinks)

Application: Shortest Distances

Low treewidth decompositions were used to compute shortest distances efficiently in graphs [Akiba et al., 2012]

- Might suggest a reason why contraction-based approaches work so well for shortest distances in infrastructure networks





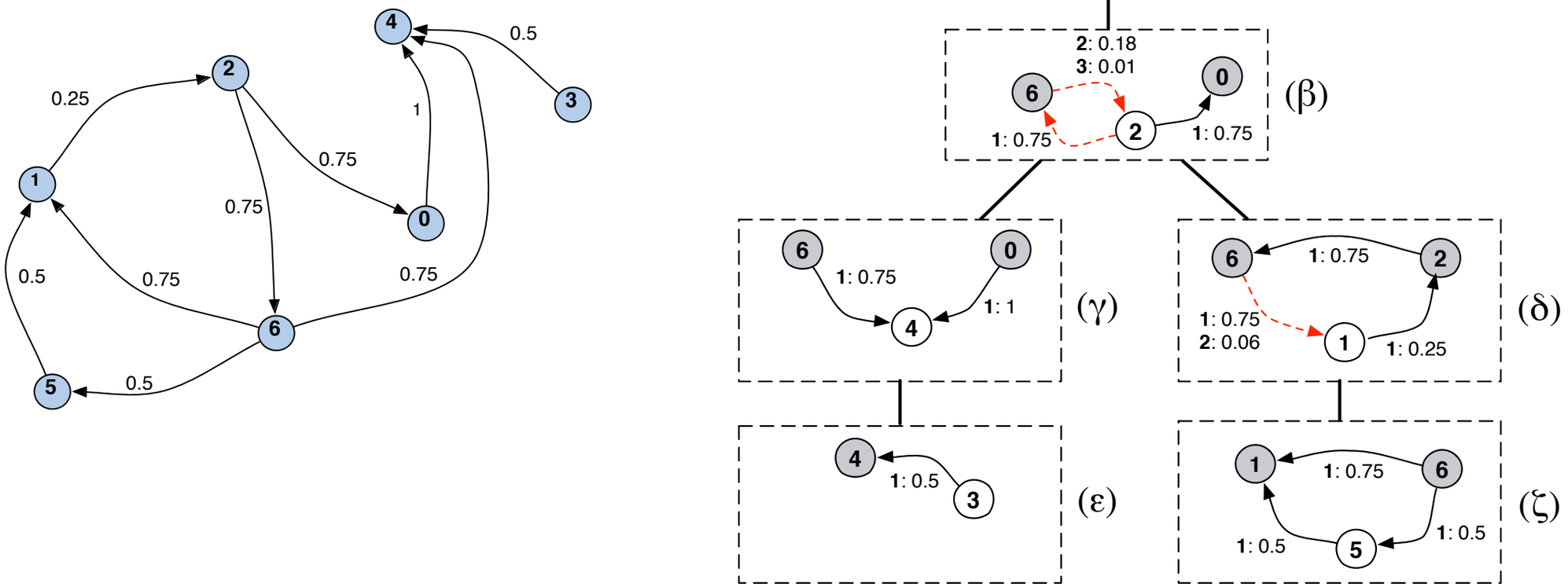
Application: Probabilistic Graphs

In our previous work [Maniu et al, 2017], we used tree decompositions to compute efficiently reachability probabilities in probabilistic graphs:

1. compute exact probabilities in low treewidth areas, and

2. combine then in the root graph and use sampling to estimate probabilities

Application: Probabilistic Graphs



[Maniu et al., 2017]

Conclusions

- Bounded-treewidth data has nice theoretical properties, but …
- … there are no graphs in the real-word having low treewidth.
- However, hope for practical applications remains when using partial tree decompositions!

Thank you!