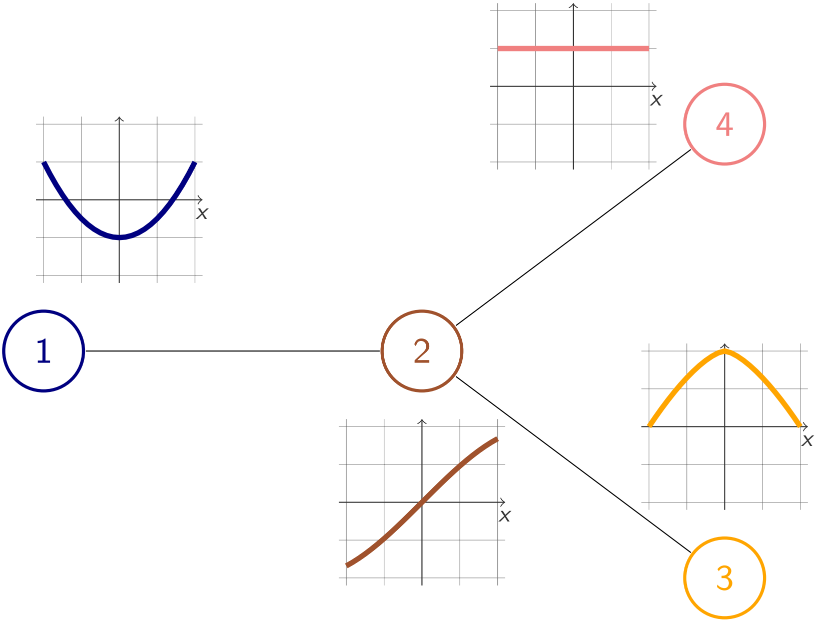
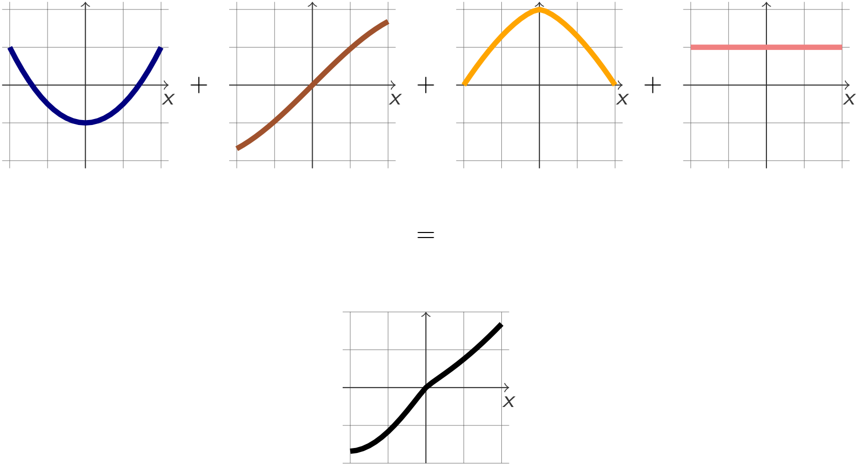
Distributed Optimization in Multiagent Systems

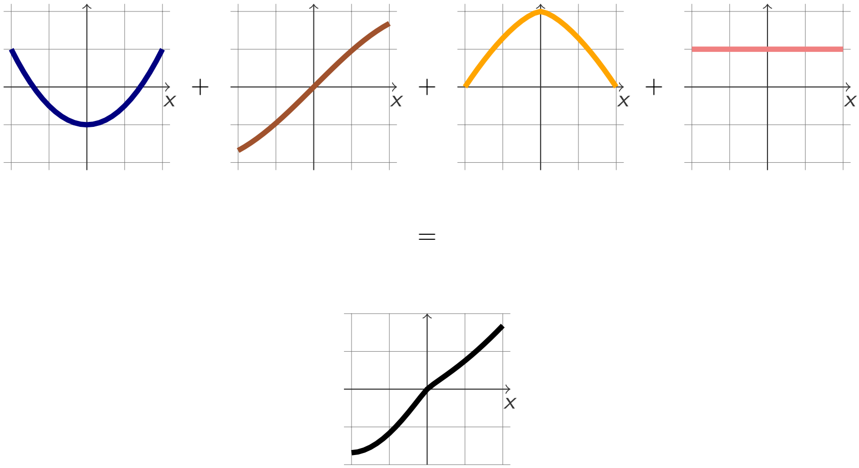
The Consensus Problem



The Consensus Problem



The Consensus Problem



No single agent knows the target function to optimize

Formally

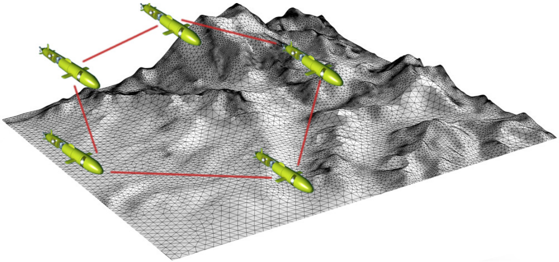
$$\min_{x \in \mathcal{X}} \sum_{n=1}^{N} f_n(x)$$

- N = number of nodes / agents
- $\mathcal{X} = \mathbb{R}^d$
- f_n is the cost function of agent n
- Two agents n and m can exchange messages if $n \sim m$

Numerous works on that problem
Early work: Tsitsiklis '84

Example #1: Wireless Sensor Networks

Y_n = random observation of sensor n
x = unknown parameter to be estimated



$$p(Y_1, \cdots, Y_N; x) = p_1(Y_1; x) \cdots p_N(Y_N; x)$$

The maximum likelihood estimate writes

$$\hat{x} = \arg\max_x \sum_n \ln p_n(Y_n; x)$$

[Schizas'08, Moura'11]

Example #2: Machine Learning

Data set formed by T samples (X_i, Y_i) $(i = 1 \dots T)$

- Y_i = variable to be explained
- X_i = explanatory features

$$\min_x \sum_{i=1}^{T} \ell(x^T X_i, Y_i) + r(x)$$

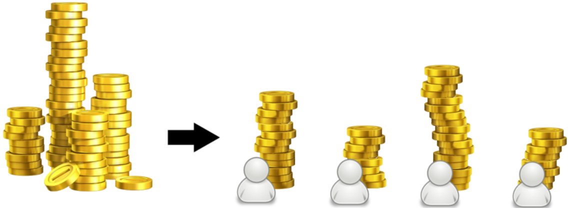
Split data into N batches

$$\min_x \sum_{n=1}^{N} \sum_i \ell(x^T X_{i,n}, Y_{i,n}) + r(x)$$

n.b.: some problems are more involved (I. Colin'16)

$$\min_x \sum_i \sum_j f(x; X_i, Y_i, X_j, Y_j) + r(x)$$

Example #3: Resource Allocation



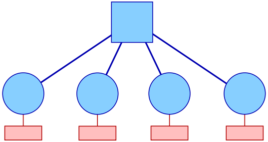
Let x_n be the resource of an agent n

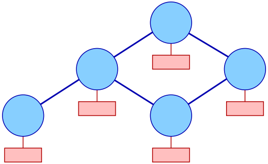
- Agents share a resource b: $\sum_n x_n \leq b$
- Agent n gets reward $R_n(x_n)$ for using resource x_n
- Maximize the global reward

$$\max_{x:\sum_n x_n \leq b} \sum_{n=1}^{N} R_n(x_n)$$

The dual of a sharing problem is a consensus problem

Networks

Parallel: 

Distributed: 

Outline

Outline

Adapt-and-combine (Tsitsiklis'84)

- `[Local step]` Each agent n generates a temporary update

$$\tilde{x}_n^{k+1} = x_n^k - \gamma_k \nabla f_n(x_n^k)$$

- `[Agreement step]` Connected agents merge their temporary estimates

$$x_n^{k+1} = \sum_{m \sim n} A(n,m)\, \tilde{x}_m^{k+1}$$

where A satisfies technical constraints (must be doubly stochastic)

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Convergence rates (e.g. [Nedic'09], [Duchi'12])

- Decreasing step size $\gamma_k \to 0$ is needed in general
- Sublinear converges rates

More problems

1. Asynchronism
   Some agents are active at time n, others aren't
   Random link failures

2. Noise
   Gradients may be observed up to a random noise (online algorithms)

3. Constraints

$$\text{Minimize } \sum_{n=1}^{N} f_n(x) \text{ subject to } x \in C$$

$$\begin{aligned} \tilde{x}_n^{k+1} &= \mathrm{proj}_C[x_n^k - \gamma_k(\nabla f_n(x_n^k) + \mathit{noise})] \\ x_n^{k+1} &= \sum_{m \sim n} A_{k+1}(n,m)\, \tilde{x}_m^{k+1} \end{aligned}$$

Distributed stochastic gradient algorithm

Under technical conditions,

Convergence (Bianchi et al.'12): x_n^k tends to a KKT point $x^\star$

Convergence rate (Morral et. al'12): If $x^\star \in \mathrm{int}(C)$

$$\sqrt{\gamma_k}^{-1}(x_n^k - x^\star) \xrightarrow{\mathcal{L}} \mathcal{N}\left(0, \Sigma_{OPT} + \Sigma_{NET}\right)$$

- Σ_{OPT} is the covariance corresponding to the centralized setting
- Σ_{NET} is the excess variance due to the distributed setting

Remark: $\Sigma_{NET} = 0$ for some protocols which can be characterized

Outline

Distributed gradient descent

Distributed Alternating Direction Method of Multipliers (D-ADMM)

Total Variation Regularization on Graphs

Alternating Direction Method of Multipliers

Consider the generic problem

$$\min_x F(x) + G(Mx)$$

where F, G are convex. Rewrite as a constrained problem

$$\min_{z=Mx} F(x) + G(z)$$

The augmented Lagrangian is:

$$\mathcal{L}_\rho(x, z; \lambda) = F(x) + G(z) + \langle \lambda, Mx - z \rangle + \frac{\rho}{2}\|Mx - z\|^2$$

ADMM

$$\begin{aligned}
x^{k+1} &= \arg\min_x \mathcal{L}_\rho(x, z^k; \lambda^k) \rightarrow \text{only } F \text{ needed} \\
z^{k+1} &= \arg\min_z \mathcal{L}_\rho(x^{k+1}, z; \lambda^k) \rightarrow \text{only } G \text{ needed} \\
\lambda^{k+1} &= \lambda^k + \rho(Mx^{k+1} - z^{k+1})
\end{aligned}$$

Back to our problem

All functions $f_n : X \to \mathbb{R}$ are assumed convex. Consider the problem:

$$\min_{u \in X} \sum_{n=1}^{N} f_n(u)$$

Main trick: Define

$$F : x = (x_1, \ldots, x_N) \mapsto \sum_n f_n(x_n)$$

Equivalent problem:

$$\min_{x \in X^N} F(x) + \iota_{\mathrm{sp}(1)}(x)$$

$$\text{where } \iota_{\mathrm{sp}(1)}(x) \quad = \quad \begin{cases} 0 & \text{if } x_1 = \cdots = x_N \\ +\infty & \text{otherwise} \end{cases}$$

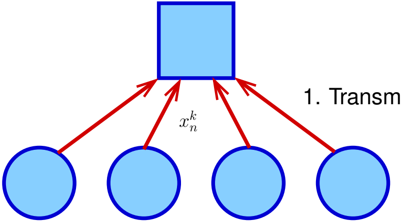
- F is separable in $x_1, \ldots, x_N$
- $G = \iota_{\mathrm{sp}(1)}$ couples the variables but is simple

ADMM illustrated

Set $\overline{x}^k = \frac{1}{N}\sum_n x_n^k$

Algorithm (see e.g. [Boyd'11])

$$\text{For all } n, \quad \begin{aligned} \lambda_n^k &= \lambda_n^{k-1} + \rho(x_n^k - \overline{x}^k) \\ x_n^{k+1} &= \arg\min_y \, f_n(y) + \frac{\rho}{2}\|\overline{x}^k - \rho^{-1}\lambda_n^k - y\|^2 \end{aligned}$$



1. Transmit current estimates

ADMM illustrated

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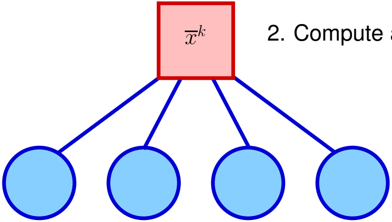

2. Compute average $\overline{x}^k$

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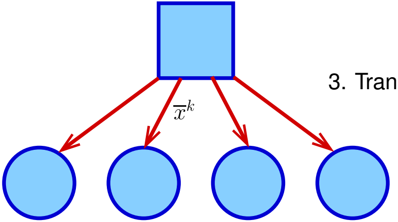

3. Transmit $\overline{x}^k$ to all agents

ADMM illustrated

Set $\overline{x}^k = \frac{1}{N}\sum_n x_n^k$

Algorithm (see e.g. [Boyd'11])

$$
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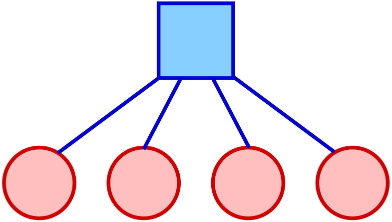

4. Compute λ_n^k, x_n^{k+1} for all n

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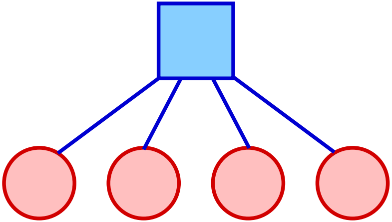
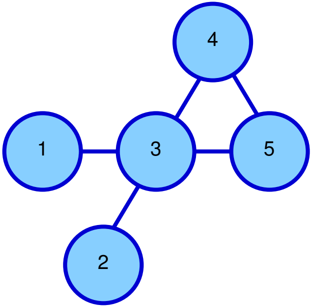


4. Compute λ_n^k, x_n^{k+1} for all n

The algorithm is *parallel* but not *distributed* on the graph

Subgraph consensus

Let $A_1, A_2, \cdots, A_L$ be subsets of agents



$A_1 = \{1, 3\}$, $A_2 = \{2, 3\}$, $A_3 = \{3, 4, 5\}$

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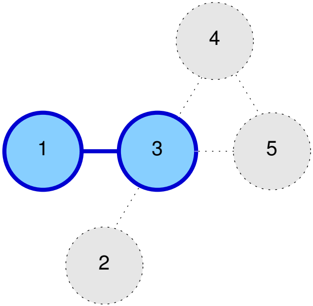

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$$\begin{pmatrix} x_1 \\ x_3 \end{pmatrix} \in \mathrm{sp} \binom{1}{1}$$

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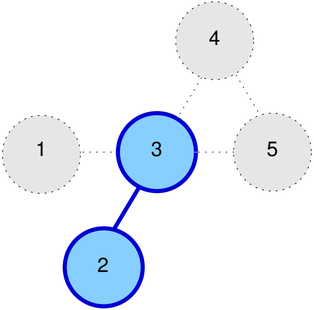

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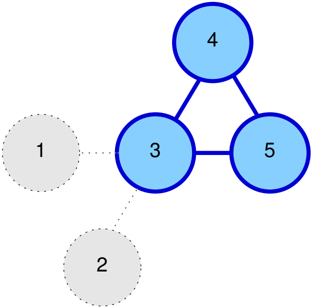

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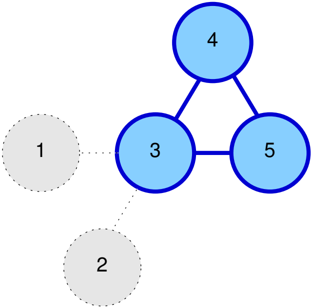

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$A_1 = \{1, 3\},\ A_2 = \{2, 3\},\ A_3 = \{3, 4, 5\}$

consensus within subgraphs $\Leftrightarrow$ global consensus

Example (Cont.)

The initial problem is

$$\min_{x \in X^N} F(x) + G(Mx)$$

$$\text{where} \quad Mx = \begin{pmatrix} x_1 \\ x_3 \\ x_2 \\ x_3 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} \quad \text{that is:} \quad M = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

and where G is the indicator function of the subspace of vectors of the form

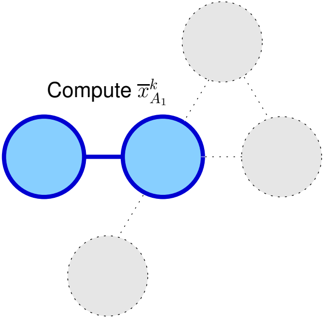
$$\begin{pmatrix} \alpha \\ \alpha \\ \beta \\ \beta \\ \delta \\ \delta \\ \delta \end{pmatrix}$$

Distributed ADMM illustrated

Distributed ADMM (early works by [Schizas'08])

$$\text{For all } n, \quad \begin{aligned} \Lambda_n^k &= \Lambda_n^{k-1} + \rho(x_n^k - \chi_n^k) \\ x_n^{k+1} &= \arg\min_y f_n(y) + \frac{\rho|\sigma_n|}{2}\|\chi_n^k - \rho^{-1}\Lambda_n^k - y\|^2 \end{aligned}$$

where $|\sigma_n|$ = number of "neighbors" of n



1. For each subgraph, compute average $\overline{x}_{A_\ell}^k$

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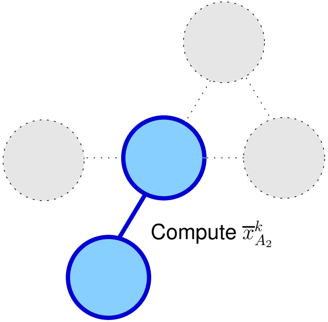

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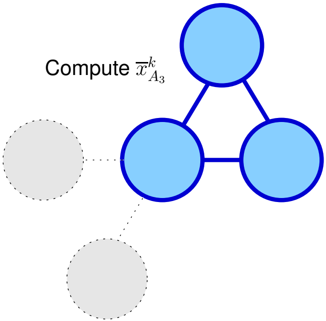

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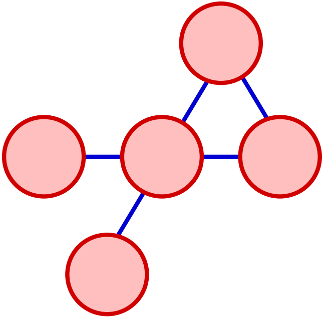

2. For each n, compute $\chi_n^k = \text{Average}(\overline{x}_{A_\ell}^k \ : \ell \text{ s.t. } n \in A_\ell)$

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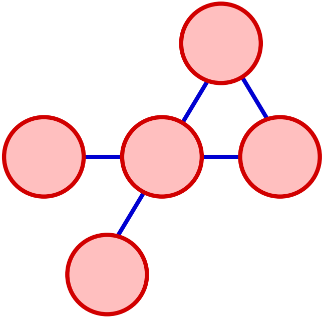


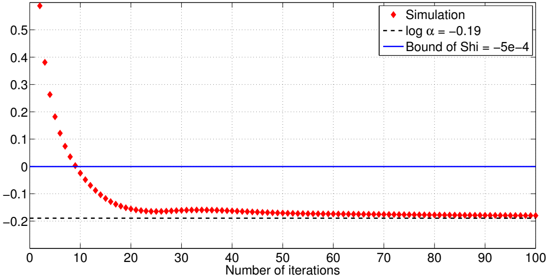
3. For each n, compute λ_n^k and x_n^{k+1}

Linear convergence of the Distributed ADMM

Assumption: $H_\star := \sum_n \nabla^2 f_n(x_\star) > 0$ at the minimizer $x_\star$

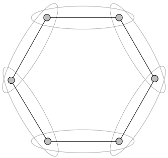
$$\|x_n^k - x_\star\| \sim \alpha^k \quad \text{as } k \to \infty$$

- [Shi et al.' 13] non-asymptotic bound but pessimistic
- [Iutzeler et al.' 16] asymptotic but tight



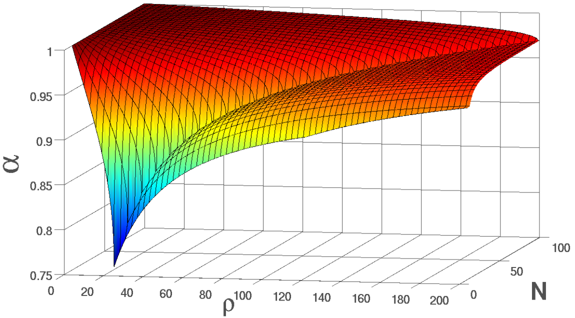
$\frac{1}{k} \log \|x^k - \mathbf{1} \otimes x_\star\|$ as a function of k

Example: ring network



Define $\alpha = \lim_{k\to\infty} \|x_n^k - x_\star\|^{1/k}$

Set $f_n : \mathbb{R} \to \mathbb{R}$ and $f_n''(x_\star) = \sigma^2$



$$\alpha \geq \sqrt{\frac{1+\cos\frac{2\pi}{N}}{2(1+\sin\frac{2\pi}{N})}} \quad \text{with equality when} \quad \rho = \frac{\sigma^2}{2\sin\frac{2\pi}{N}}$$

Asynchronous D-ADMM

- All agents must complete their $\arg\min$ computation before combining
- The network waits for the slowest agents

Our objective: allow for asynchronism

Revisiting ADMM as a fixed point algorithm

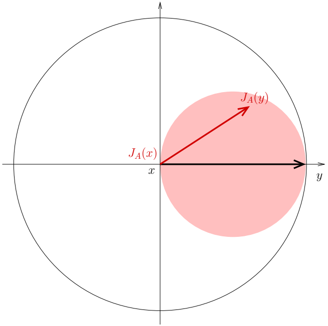
Set $\zeta^k = \lambda^k + \rho z^k$. Fact: $\lambda^k = P(\zeta^k)$ where P is a projection.

ADMM can be written as a fixed point algorithm [Gabay,83] [Eckstein,92]

$$\zeta^{k+1} = J(\zeta^k)$$

where J is firmly non-expansive *i.e.*,

$$\|J(x) - J(y)\|^2 \leq \|x - y\|^2 - \|(I - J)(x) - (I - J)(y)\|^2$$



Random coordinate descent

Introducing the block-components of $\zeta^{k+1} = J(\zeta^k)$:

$$\begin{pmatrix} \zeta_1^{k+1} \\ \vdots \\ \zeta_\ell^{k+1} \\ \vdots \\ \zeta_L^{k+1} \end{pmatrix} = \begin{pmatrix} J_1(\zeta^k) \\ \vdots \\ J_\ell(\zeta^k) \\ \vdots \\ J_L(\zeta^k) \end{pmatrix}$$

Random coordinate descent

If only one block $\ell = \ell(k+1)$ is active at time $k+1$:

$$\begin{pmatrix} \zeta_1^{k+1} \\ \vdots \\ \zeta_\ell^{k+1} \\ \vdots \\ \zeta_L^{k+1} \end{pmatrix} = \begin{pmatrix} \zeta_1^{k} \\ \vdots \\ J_\ell(\zeta^k) \\ \vdots \\ \zeta_L^{k} \end{pmatrix}$$

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Convergence of the Asynchronous ADMM [Iutzeler'13]

This algorithm still converges if active components are chosen at random

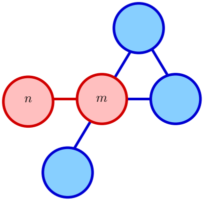
Main idea: For a well-chosen norm $\|\!|\,.\,|\!\|$ and a fixed point $\zeta^\star$ of J, prove

$$\mathbb{E}\left(\left|\!\left|\!\left| \zeta^{k+1} - \zeta^\star \right|\!\right|\!\right|^2 | \mathcal{F}_k \right) \leq \left|\!\left|\!\left| \zeta^{k} - \zeta^\star \right|\!\right|\!\right|^2$$

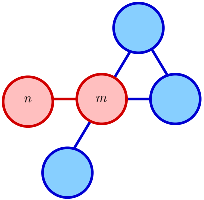
$\Rightarrow \zeta^k$ is getting "stochastically" closer to $\zeta^\star$

Asynchronous ADMM explicited

Activate two nodes $A_\ell = \{m, n\}$



Asynchronous ADMM explicited



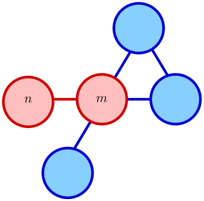
Activate two nodes $A_\ell = \{m, n\}$

- Agent n computes

$$x_n^{k+1} = \arg\min_x f_n(x) + \sum_{j\sim k} \left(\langle x, \lambda_{j,n}^k \rangle + \frac{\rho}{2} \|x - \bar{x}_{j,n}^k\|^2 \right)$$

and similarly for Agent m.

Asynchronous ADMM explicited



Activate two nodes $A_\ell = \{m, n\}$

- Agent n computes

$$x_n^{k+1} = \arg\min_x f_n(x) + \sum_{j\sim k} \left(\langle x, \lambda_{j,n}^k \rangle + \frac{\rho}{2} \|x - \bar{x}_{j,n}^k\|^2 \right)$$

  and similarly for Agent m.

- They exchange x_m^{k+1} and x_n^{k+1}
- Agent n computes

$$\bar{x}_{m,n}^{k+1} = \frac{1}{2}(x_m^{k+1} + x_n^{k+1}),$$

$$\lambda_{m,n}^{k+1} = \lambda_{m,n}^k + \rho \frac{x_n^{k+1} - x_m^{k+1}}{2}$$

  and similarly for Agent m.

Generalization: Distributed Vũ-Condat algorithm

- Vũ-Condat algorithm generalizes ADMM (allows "gradients" evaluations)
- Distributed Vũ-Condat algorithm is applicable using the same principle
- Bianchi'16, Fercoq'17 provide a random coordinate descent version
- The algorithm is asynchronous at the *node level* and not at the *edge level*

Stochastic Optimization

$$\min_{x \in \mathcal{X}} \sum_{n=1}^{N} \mathbb{E}\left(f_n(x, \xi_n)\right)$$

- Law of ξ_n unknown, but revealed *on-line* through random copies $\xi_n^1, \xi_n^2, \ldots$
- **Stochastic approximation**: at time k, replace the unknown function $\mathbb{E}\left(f_n(\,.\,, \xi_n)\right)$ by its random version $f_n(\,.\,, \xi_n^k)$

  Example : stochastic gradient descent
- **Thesis of A. Salim**: Stochastic versions of generic optimization algorithms (Forward-Backward, Douglas-Rachford, ADMM, Vũ-Condat, etc.)
- Byproduct : distributed stochastic algorithms

Outline

Distributed gradient descent

Distributed Alternating Direction Method of Multipliers (D-ADMM)

Total Variation Regularization on Graphs

Total variation regularization (1/2)

Notation: On a graph $G = (V, E)$, the total variation of $x \in \mathbb{R}^V$ is

$$\mathrm{TV}(x) = \sum_{\{i,j\} \in E} |x_i - x_j|$$

General problem:

$$\min_{x \in \mathbb{R}^V} F(x) + \mathrm{TV}(x)$$

- Trend filtering: $F(x) = \frac{1}{2}\|x - m\|^2$ where $m \in \mathbb{R}^V$ are noisy measurements
- Graph inpainting: complete possibly missing measurements on the nodes

Proximal gradient algorithm:

$$x_{n+1} = \mathrm{prox}_{\gamma \mathrm{TV}}(x_n - \gamma \nabla F(x_n))$$

- Computing $\mathrm{prox}_{\mathrm{TV}}$ is difficult over large unstructured graphs
- But efficient algorithms exist for 1D-graphs (Mammen'97) (Condat'13)

Total variation regularization (2/2)

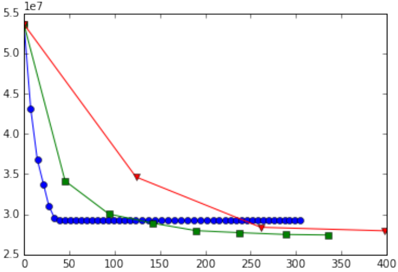
Write TV as an expectation: Let ξ be simple random walk in G of fixed length

$$\mathrm{TV}_G(x) \;\propto\; \mathbb{E}(\mathrm{TV}_\xi(x))$$

Algorithm (Salim'16) At time n,

- Draw a random walk ξ_{n+1}
- Compute $x_{n+1} = \mathrm{prox}_{\gamma_n \mathrm{TV}_{\xi_{n+1}}}(x_n - \gamma_n \nabla F(x_n)) \to$ easy, 1D

Hidden difficulty: one should avoid loops when choosing the walk...



Trend filtering example. Cost function vs time(s). Stochastic block model 10^5 nodes, 25.10^6 edges.

Blue: Stochastic proximal gradient, Green: dual proximal gradient, Red: dual L-BFGS-B