Labeled variant of Dilworth's theorem

Antoine Amarilli, Charles Paperman

This note presents a generalization of Dilworth's theorem [1] to labeled posets. As it turns out, we were only able to obtain results for the case of the alphabet $\{a, b\}$, i.e., an alphabet with two elements. This note is work-in-progress and has not been proofread carefully, so the results should be taken with a grain of salt: caveat lector.

1 Introduction

We consider a partial order $(G, <)$, which we equivalently see as a transitive DAG. An *antichain* of G is a subset of vertices of G that are pairwise incomparable, and the *width* of G is the cardinality of its largest antichain. Dilworth's theorem states that the width is equal to the minimal cardinality of a *chain partition* of G, i.e., a partition $G = G_1 \sqcup \cdots \sqcup G_n$ such that the restriction of $<$ to each G_i is a total order.

We generalize this result to *labeled posets*, i.e., we fix a non-empty alphabet A, and each vertex x in G carries a label $\lambda(x) \in A$. For any set Y of vertices and $a \in A$, we write $|Y|_a$ to mean $|\{y \in Y \mid \lambda(y) = a\}|$. For non-empty $A' \subseteq A$, the A'-*size* of a subset $Y \subseteq G$ is $\min_{a \in A'} |Y|_a$. The A'-*width* of G is the maximal A'-size of an antichain of G.

We fix a *width threshold* $k \in \mathbb{N}_{>0}$. For $A' \subseteq A$, we call A' *frequent* in G for k if G has A'-width at least k, and call it *rare* otherwise. The *spectrum* of G for k is the function f mapping each non-empty $A' \subseteq A$ to 1 if A' is rare and 0 if it is frequent. It is clear that f is a monotone Boolean function, i.e., if $A' \subseteq A''$ and A' is rare in G for k, then A'' is also rare, because any antichain of A''-size $\geq k$ is in particular an A'-antichain of A'-size $\geq k$. In particular, if every singleton set is rare, this means that there is no antichain of $\{a\}$-size k for any $a \in A$, which implies that the width of G (in the standard sense) is at most $k \times |A|$: this is the "most constrained" case. Conversely, if A is frequent, this means that there is an antichain containing k copies of each possible letter as we want, which is the "least constrained" case.

Conversely, almost any monotone Boolean function f can be realized as a spectrum of some DAG G, e.g., the one built as a serial composition of one antichain for each A' which is frequent according to f, that consists of k vertices labeled a for each $a \in A'$ (note that this is non-empty).

Our goal is to answer the following question: knowing the spectrum of a DAG, what can we tell about its structure? Dilworth's theorem can be seen as the case where

$A = \{a\}$: either A is frequent and G has unbounded width, or A is rare and G has bounded width.

2 Case of $A = \{a, b\}$

We start by studying the simpler case of an alphabet with two letters only. The least constrained spectrum is the one where $\{a, b\}$ is frequent, meaning that there is an antichain containing k elements labeled a and k-elements labeled b, and we cannot hope to say anything more interesting here. The most constrained spectrum is the one where $\{a\}$ and $\{b\}$ are both rare, meaning that the width is globally bounded. Another uninteresting possibility is when $\{a\}$ is the only frequent subalphabet, which means that G has no large antichain of b elements; so we can look at the restriction of G to b-labeled elements, say that it has width bounded by k, and that's all. There is obviously another uninteresting symmetric case where $\{b\}$ is the only frequent subalphabet. However, there is an interesting case: the spectrum where $\{a\}$ and $\{b\}$ are both frequent but $\{a, b\}$ is infrequent: this is the case that we will study.

Remember that one possible scenario for this is when the DAG is a series composition of a part with a large antichain of a-labeled elements, and a part with a large antichain of b-labeled elements. We will show that the graph can be decomposed in a similar way.

A *convex set* of a DAG G is a subset X of its vertices such that, for any vertices $u \leq v \leq w$ of G, if $u \in X$ and $w \in X$ then $v \in X$. A *layering* of a DAG G is a sequence $L_1, \ldots, L_n$ of convex sets of G, called *layers*, that are a partition of the vertices of G, such that, for any $u \leq v$ in G, letting L_i and L_j be the respective layers of u and v, we have $i \leq j$.

In a layering, the order relation across layers is unspecified, i.e., it is not necessarily total, unlike a series composition. However, in our case, the composition will be "almost serial". We formalize this by saying that the layering is *discriminative*. For $k \in \mathbb{N}$, we say that a layering $L_1, \ldots, L_n$ of a DAG G is *k-discriminative* if, for every antichain A of G, there is a layer L_i such that A is "almost contained" in L_i, formally $|A \setminus L_i| \leq k$. Note that this requirement imposes no condition on antichains of G of size $\leq k$.

We will show the following claim:

Theorem 2.1. *For any constant $k \in \mathbb{N}_{>0}$, given an $\{a, b\}$-DAG G where $\{a, b\}$ is rare for k but $\{a\}$ and $\{b\}$ are frequent for k, we can compute in PTIME a $15k$-discriminative layering $L_1, \ldots, L_n$ such that, for every $i \in \mathbb{N}$, either $\{a\}$ is rare for $15k$ and $\{b\}$ is frequent for $2k$ in L_i, or $\{b\}$ is rare for $15k$ and $\{a\}$ is frequent for $2k$ in L_i.*

To show this result, we will start by a simple layering construction on G:

Lemma 2.2. *We can determine in PTIME whether G has width $\geq 3k$, and if it does we can compute in PTIME a layering $L_1, \ldots, L_n$ such that, for all $1 \leq i \leq n$, the layer L_i has width $< 6k$, and one of $\{a\}$ and $\{b\}$ is frequent in L_i for threshold $2k$.*

To prove this lemma, we introduce an auxiliary notion on antichains. We will say that an antichain X is *above* an antichain X', written $X \leq X'$, if X is included in the union of the ancestors of the elements of X', formally, for each $x \in X$, there exists $x' \in X'$ such that $x \leq x'$.

Lemma 2.3. *The relation $\leq$ is an order relation.*

Proof. We first show that $\leq$ is transitive. Assume that $X \leq X'$ and $X' \leq X''$. Then, for every $x \in X$, there exists $x' \in X'$ such that $x \leq x'$, and for this x' there exists $x'' \in X''$ such that $x' \leq x''$. By transitivity we conclude that $x \leq x''$. Hence, it is indeed the case that for every $x \in X$ there is $x'' \in X''$ such that $x \leq x''$, so we have $X \leq X''$

Second, we show that $\leq$ is antisymmetric. Assume that $X \leq X'$ and $X' \leq X$, and assume by way of contradiction that $X \neq X'$. We assume without loss of generality that $X \setminus X'$ is not empty. Take $x \in X \setminus X'$ and consider its witnessing element $x \leq x'$ with $x' \in X'$. Observe that necessarily $x < x'$ because $x' \in X'$ but $x \notin X'$. Now, consider the element $x'' \in X$ of x' such that we have $x' \leq x''$. We deduce by transitivity that $x < x''$, which contradicts the fact that X is an antichain. □

We can now prove Lemma 2.2:

Proof. We will work by induction on the number of vertices of G.

We can check whether G has width $\geq 3k$ by checking whether it has an antichain of size $3k$, in time $O(|G|^{3k})$, hence in PTIME. If it does not, there is nothing more to show. If it does, then pick some antichain X of G of size $3k$ which is minimal in the order $\leq$: this can be done naively in PTIME by computing explicitly the relation $\leq$ on antichains of size $3k$. As $\{a, b\}$ is rare for threshold k in G, we know that the $\{a, b\}$-width of X is $< k$, so that either $|X|_b < k$ or $|X|_a < k$. We will assume the first case, the second being symmetric; so in particular we know that $|X|_a > 2k$. Let U be the union of the ancestors of X (including X): we know that U has width $\leq 3k$, because if U contains an antichain Y of cardinality $3k + 1$ then there is a subset Y' of Y of size $3k$ which is different from X, and by definition $Y' < X$, contradicting the minimality of X. Let G' be the restriction of G to the complement of U.

We now use the induction hypothesis to process G' recursively. If G' has width $< 3k$, then we decompose G in the singleton layer $L_1 := G$. In this first case, we know that a is frequent in L_1 for threshold $> 2k$, because L_1 contains the antichain X and $|X| = 3k$ and $|X|_b < k$ so $|X|_a > 2k$. Further, we know that L_1 has width $< 6k$ because it is the union of U which has width $\leq 3k$ and G' which has width $< 3k$.

If G' has width $\geq 3k$, we let $L'_1, \ldots, L'_{n'}$ be the layering of G' obtained by induction, and take our layering of G to be $U, L'_1, \ldots, L'_{n'}$ of G. We know that U has width $\leq 3k$, hence width $< 6k$; we know that $\{a\}$ is frequent for threshold $2k$ in U because it contains U; and we know by induction that the layers $L'_1, \ldots, L'_{n'}$ satisfy our conditions. This concludes the proof. □

We now show a useful result on antichains and layerings, called the *antichain shuffling lemma*:

Lemma 2.4. *Let $L_1, \ldots, L_n$ be a layering of G, and let X and Y be two disjoint antichains such that $X \subseteq L_i$ for some $1 \leq i \leq n$. Define $Y_- := Y \cap \bigcup_{i'<i} L_i$ and*

$Y_+ := Y \cap \bigcup_{i'>i} L_i$. *Then there are* $X' \subseteq X$ *and* $Y' \subseteq Y$ *such that* $|X'| \geq |X|/2$, *such that* $|Y'| \geq \min(|Y_-|, |Y_+|)$, *and such that* $X' \cup Y'$ *is an antichain.*

Proof. Let $X_+ \subseteq X$ be the elements $x \in X$ such that there exists a $y \in Y_+$ such that $x \leq y_+$, and let $X_- \subseteq X$ be the elements $x \in X$ such that there exists a $y \in Y_-$ such that $y_- \leq x$. It is clear that X_+ and X_- must be disjoint, because if there existed $x \in X_- \cap X_+$ then, taking two witnessing y_- and y_+, by transitivity we would have $y_- \leq y_+$, but as Y_- and Y_+ are disjoint we have $y_- < y_+$, contradicting the fact that Y is an antichain. We assume without loss of generality that $|X_-| \geq |X_+|$, the other case is symmetric. Let $X' := X \setminus X_+$, we know that $|X'| \geq |X|/2$. We now argue that $Y' := Y_+$ satisfies the statement. Indeed, the bound on the cardinality of Y' holds. Now, assume by way of contradiction that there are $x \in X'$ and $y \in Y'$ that are comparable. We cannot have $x \leq y$, as otherwise, since $y \in Y_+$, we would have $x \in X_+$, so by definition we cannot have $x \in X'$. Further, we cannot have $x \geq y$ because x is in layer L_i but y is in Y_+ so it is in a layer $L_{i'}$ with $i' > i$, so this would contradict the definition of a layering. Hence, indeed, $X' \cup Y'$ is an antichain. □

We can now use Lemma 2.2 and the antichain shuffling lemma to show Theorem 2.1:

Proof. We start with the layering $L'_1, \ldots, L'_{n'}$ obtained in Lemma 2.2. Recall that, by the lemma statement, for each $1 \leq i \leq n'$ there is $x'_i \in \{a, b\}$ such that x'_i is frequent in L'_i; if both letters are frequent then we make an arbitrary choice for x'_i. We now define a new layering $L_1, \ldots, L_n$ by merging together the consecutive L'_i that have the same value for x'_i; for each $1 \leq j \leq n$ we write x_j the common value of x'_i for the L'_i user to create L_j. It is clear that $L_1, \ldots, L_n$ is still a layering, and that it is constructed in PTIME; we now show that it satisfies the conditions of the theorem. First, it is clear that for every $1 \leq i \leq n$, the letter x_j is frequent for threshold $2k$ in L_j, because this is the case of the L'_i used to create L_j. Second, we must show that the other letter is rare, and that the layering is discriminative. To do this, we will show the following auxiliary claim: (*) for any antichain Y of G, letting x be the most common letter in Y, there is $1 \leq i \leq n$ such that $|Y \setminus L_i| \leq 15k$ and $x_i = x$.

Observe first that claim (*) implies what we want to show. Indeed, it clearly implies that the layering is $15k$-discriminative (if we forget about the additional condition on x_i). Second, it implies that, for every L_i with $1 \leq i \leq n$, letting $x := x_i$ and y be the other letter, then y is $15k$-rare in L_i. Indeed, for any antichain Y of y-labeled elements in L_i, as y is the most common element of Y and Y is also an antichain of G, we know by (*) that $|Y \setminus L_j| \leq 15k$ for some $1 \leq j \leq n$ with $x_j = y$, which implies $j \neq i$, so Y and L_j are disjoint and so we know that $|Y| \leq 15k$. Hence, all that remains is to show claim (*).

To show claim (*), let Y_0 be an antichain of G. We assume that $|Y_0| > 15k$, as there is nothing to show otherwise. Let x be the most common letter in Y, and let y be the other letter. Let $Y \subseteq Y_0$ be the subset of elements labeled x in Y; as $\{a, b\}$ is rare for threshold k in G, we know that $|Y_0|_y < k$, so that $|Y| > 14k$.

We will write for simplicity $L^{\uparrow}(i) := \bigcup_{i'<i} L'_{i'}$ for $1 \leq i \leq n' + 1$, and $L^{\downarrow}(i) := \bigcup_{i'>i} L'_{i'}$ for $0 \leq i \leq n'$; and we write $Y^{\uparrow}(i) := Y \cap L^{\uparrow}(i)$, write $Y^{\downarrow}(i) := Y \cap L^{\downarrow}(i)$, and write $Y^{=}(i) := Y \cap L_i$ Let us now define a function $g^{\uparrow}$ mapping each $i \in \{1, \ldots, n' + 1\}$ to

$\left|Y^{\uparrow}(i)\right|$: this function is nondecreasing, we have $g^{\uparrow}(1) = 0$, and $g^{\uparrow}(n'+1) = |Y|$. There are two cases: either there is $1 \leq i_0 \leq n$ such that $x'_{i_0} = y$ and $g^{\uparrow}(i_0) \geq k$, or there is no such i_0.

Case 1: there is no such i_0. In this case, there are two subcases. The first subcase is when there are no layers L'_i at all such that $x'_i = y$, i.e., we have $n = 1$, and the only layer L_1 is such that $x'_1 = x$; but in this case claim (*) holds because we can simply take $i := 1$. The second subcase is where there are layers such that $x'_i = y$; let i_1 be the largest i such that $x'_{i_1} = y$. We know that all L'_i with $i > i_1$ are such that $x'_i = x$ (by our assumption on the inexistence of i_0), so they are all merged into the last layer L_n. Now, we know that $\left|Y^{\uparrow}(i_1)\right| < k$, we know that $|Y^{=}(i_1)| < 6k$ because L'_{i_1} has width $< 6k$, so we know that all elements of Y except at most $7k$ are in L_n; hence, as $|Y_0 \setminus Y| < k$, all elements of Y_0 except at most $8k$ are in L_n; this shows claim (*) in case 1.

Case 2: there is i_0 such that $x'_{i_0} \neq x$ and $g^{\uparrow}(i_0) \geq k$. In this case, we take for i_0 the smallest possible value such that this holds; of course we know that $i_0 > 1$ as $g^{\uparrow}(1) = 0$. We now define $g^{\downarrow}(i_0) := \left|Y^{\downarrow}(i)\right|$, and show that we must have $g^{\downarrow}(i_0) < k$. Indeed, if $g^{\downarrow}(i_0) \geq k$, then we can consider the antichain X of $2k$ a-labeled elements which is known to exist in L'_{i_0}, and we can use the antichain shuffling lemma (Lemma 2.4) to conclude that there is $X' \subseteq X$ with $|X'| \geq k$ and $Y' \subseteq Y$ with $|Y'| \geq \min(g^{\uparrow}(i_0), g^{\downarrow}(i_0)) \geq k$ such that $X' \cup Y$ is an antichain, but this is impossible because it contains $|X'| \geq k$ elements labeled a and $|Y'| \geq k$ elements labeled b, contradicting the assumption that $\{a, b\}$ is rare for threshold k in G. So indeed we have $g^{\downarrow}(i_0) < k$. We now distinguish two subcases: either there is $i_1 < i_0$ such that $x'_{i_1} = y$ or there is none.

The first subcase is when no such i_1 exists. Then we know that, for $1 \leq i' < i_0$, each layer $L'_{i'}$ is such that $x'_{i'} = y$, so they are merged together in the first layer L_1; now as $|Y^{=}(i_0)| < 6k$ because L_{i_0} has width $< 6k$ and as $g^{\downarrow}(i_0) < k$ we know that all elements of Y except at most $7k$ are in L_1; so all elements of Y_0 except at most $8k$ are in L_1, which shows claim (*). The second subcase is when such an i_1 exists: in this case, we let i_1 be the largest value such that $1 \leq i_1 \leq i_0$ and $x'_{i_1} = y$. By minimality of i_0, we know that $g^{\uparrow}(i_1) < k$. So we know that $\left|Y^{\uparrow}(i_1)\right| < k$ and that $\left|Y^{\downarrow}(i_0)\right| < k$, and we know that $|Y^{=}(i_0)| < 6k$ and $|Y^{=}(i_1)| < 6k$ again from the width bound. As $|Y| > 14k$, this implies that we must have $i_0 - i_1 > 1$. So let us consider the layers $L'_{i_1+1}, \ldots, L'_{i_0-1}$; by maximality of i_1 we know that $x_{i'} = x$ for all $i_1 < i' < i_0$, so all these layers are merged in some layer L_j. From the inequalities above, we know that all elements of Y are in L_j except $< k$ that are in $Y^{\uparrow}(i_1)$, except $< k$ that are in $Y^{\downarrow}(i_0)$, except $< 6k$ that are in L'_{i_0}, and except $< 6k$ that are in L'_{i_1}, i.e., $14k$ exceptions at most. So all elements of Y_0 except at most $15k$ are in L_j, and indeed we have $x_j = x$. This establishes claim (*) and concludes the proof. □

Open problem. We do not know whether we replace PTIME by NL in Theorem 2.1, i.e., show that the layering can be implicitly computed in NL.

3 General alphabets

There is an annoying counter-example: consider the parallel composition of a large antichain of a and of the serial composition of a large antichain of b and of a large antichain of c. Then by discriminativity you want the a's and the b's to be in the same layer, and ditto for the a's and the c's, so everyone is in the same layer, but then the layer contains large antichains of a's, of b's, and of c's, and of $\{a, b\}$'s, and of $\{a, c\}$'s, so it is not more informative than the original instance... even though it was interesting that $\{b, c\}$ was infrequent in the instance.

References

[1] R. P. Dilworth. A decomposition theorem for partially ordered sets. *Annals of Mathematics*, 1950.