

INF280

Graph Traversals & Paths

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Contents

Introduction

Simple Traversals

Depth-First Search

Breadth-First Search

Finding Paths

Dijkstra

Bellman-Ford

Floyd-Warshall

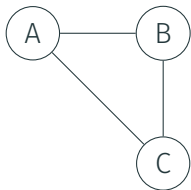
Improvements

Eulerian Circuits

Introduction

You all know graphs:

- Set of nodes N
- Set of edges $E \subseteq N \times N$
- Edges can be undirected or directed, i.e., $(a, b) \neq (b, a)$



$$N \quad \{A, B, C\}$$

$$E \quad \{(A, B), (A, C), (B, C)\}$$

Data Structures

Several options to represent graphs:

- Adjacency matrix:
 - `bool G[MAXN][MAXN];`
 - `G[x][y]` is `true` if an edge between node `x` and `y` exists
 - Replace `bool` by `int` to represent weighted edges
- Adjacency list:
 - `vector<int> Adj[MAXN];`
 - `y` is in `Adj[x]` if an edge between node `x` and `y` exists
 - Pairs to represent weights
- Edge list:
 - `vector<pair<int, int> > Edges;`
 - `Edges` contains a pair of nodes if an edge exists between them
- Nodes and edges may also be custom structs or classes

Contents

Introduction

Simple Traversals

Depth-First Search

Breadth-First Search

Finding Paths

Dijkstra

Bellman-Ford

Floyd-Warshall

Improvements

Eulerian Circuits

Depth-First Search

Visit each node in the graph once:

- Recursive implementation below
- Manage stack yourself for iterative version
- First visit child nodes then siblings

```
bool Visited[MAXN] = {};  
void DFS(int u) {  
    if (Visited[u])  
        return;  
    Visited[u] = true;  
    // maybe do something with u (pre-order) ...  
    for (auto v : Adj[u])  
        DFS(v);  
    // or do something here (post-order)  
}
```

Applications of DFS

- Determine a topological order of nodes
 - Only works for graphs without cycles (i.e., $x \rightarrow y \rightarrow z \rightarrow x$)
 - Add visited node at the head of an ordered list
(at the end of DFS: `ordering.push_front(u)`)
- Detect if a cycle exists:
 - Check if the currently visited node is on the stack
 - A) Use three states for `visited` array:
`UNVISITED, ONSTACK, VISITED`
 - B) Explicitly search in the stack of the iterative algorithm
- Examples: <https://visualgo.net/dfsbfbs>

Breadth-First Search

Visit each node in the graph once:

- Similar to DFS, but replaces **stack** by **queue**

```
queue<int> Q;
bool Visited[MAXN] = {};
void BFS(int root) {
    Q.push(root);
    while (!Q.empty()) {
        int u = Q.front();
        Q.pop();
        if (Visited[u])
            continue;
        Visited[u] = true;
        for (auto v : Adj[u])
            Q.push(v);          // usually do something with v
    }
}
```


Applications of BFS

- Shortest path search
 - Stop processing when a given node d was found
 - Minimizes number of hops, i.e., all edges have same weight
 - Generalization follows next
- Examples: <https://visualgo.net/dfsbf>

Contents

Introduction

Simple Traversals

- Depth-First Search

- Breadth-First Search

Finding Paths

- Dijkstra

- Bellman-Ford

- Floyd-Warshall

- Improvements

Eulerian Circuits

BFS can only be used if edge weights are uniform

- Dijkstra's algorithm generalizes this
- Constraint: all edges need to have non-negative weights
- Use a priority queue to choose which node to examine next
 - Would require a function to **update** the priority of an element
 - Would need to update order in the priority queue
 - We'll use the standard **priority_queue** of STL
 - Simply insert a new element in the queue (no update)
 - Ok since priorities decrease monotonically
 - This slightly diverges from Dijkstra's algorithm
- May revisit nodes several times

Dijkstra's Algorithm

```
unsigned int Dist[MAXN];
typedef pair<unsigned int, int> WeightNode;           // weight goes first
priority_queue<WeightNode, std::vector<WeightNode>,
              std::greater<WeightNode> > Q;

void Dijkstra(int root) {
    fill_n(Dist, MAXN, MAXLEN);
    Dist[root] = 0;
    Q.push(make_pair(0, root));
    while (!Q.empty()) {
        int u = Q.top().second;                       // get node with least priority
        Q.pop();
        for (auto tmp : Adj[u]) {
            int v = tmp.second;
            unsigned int weight = tmp.first;
            if (Dist[v] > Dist[u] + weight) {         // shorter path found?
                Dist[v] = Dist[u] + weight;
                Q.push(make_pair(Dist[v], v));        // simply push, no update here
            }
        }
    }
}
```

<https://visualgo.net/sssp>

- Dijkstra's algorithm is limited to non-negative edge weights
- Bellman-Ford extends this to general edge weights
- Constraint: no cycle with negative total costs
- May again revisit nodes several times

Bellman-Ford Algorithm

```
unsigned int Dist[MAXN];  
void BellmanFord(int root) {  
    fill_n(Dist, MAXN, MAXLEN);  
    Dist[root] = 0;  
    for(int u=0; u < N - 1; u++) { // just iterate N - 1 times  
        for (auto tmp : Edges) {  
            unsigned int weight = get<0>(tmp);  
            int u = get<1>(tmp); // similar to Dijkstra before  
            int v = get<2>(tmp);  
            Dist[v] = min(Dist[v], Dist[u] + weight);  
        }  
    }  
}
```

<https://visualgo.net/sssp>

Floyd-Warshall

- Dijkstra and Bellman-Ford compute shortest paths
 - From a single source (`root`)
 - To all other (reachable) nodes
 - This is known as: single-source shortest path problem
- Floyd-Warshall extends this to compute the shortest paths between **all pairs** of nodes
- This is known as: all-pairs shortest path problem

Floyd-Warshall Algorithm

```
int Dist[MAXN][MAXN];
void FloydWarshall() {
    fill_n((int*)Dist, MAXN*MAXN, MAXLEN);
    for(int u=0; u < N; u++) {
        Dist[u][u] = 0;
        for (auto tmp : Adj[u])
            Dist[u][tmp.second] = tmp.first;
    }
    for(int k=0; k < N; k++)           // check sub-path combinations
        for(int i=0; i < N; i++)
            for(int j=0; j < N; j++)   // concatenate paths
                Dist[i][j] = min(Dist[i][j], Dist[i][k] + Dist[k][j]);
}
```


Keeping track of the path

We only considered the length of the path so far:

- All of the above algorithms can track the actual shortest path
- This allows to *print* each edge/node along the path
- Basic idea:
 - Introduce an array `int Predecessor [MAXN]`
(Use two-dimensional array for Floyd-Warshall)
 - Updated whenever `Dist [v]` changes
 - Simply set to the new predecessor `u`

Heuristics – A* Search

Heuristics may speed-up the path search

- Bellman-Ford and Floyd-Warshall equally explore all possibilities
- Dijkstra *prefers* nodes with shorter distance
- Basic idea behind A* Search:
 - Extension to Dijkstra's algorithm
 - Introduce an estimation of the remaining distance
 - Prefer nodes with minimal estimated *remaining* distance
- Advantages
 - May converge faster than Dijkstra
 - Can be used to compute approximate solutions (trading speed for precision)

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Introduction

Simple Traversals

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- Breadth-First Search

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Eulerian Circuits

Eulerian Circuits

We study **undirected** graphs and assume they are **connected**:

- Eulerian path:
Use every edge of a graph **exactly** once. Start and end may **differ**
- Eulerian circuit:
Use every edge **exactly** once. Start and end at the **same node**
- Conditions to find Eulerian **path**:
 - All nodes have even degree **or**
 - Precisely two nodes have odd degree
- For Eulerian **circuit**, all nodes must have even degree

Hierholzer's Algorithm for Eulerian Paths (assuming they exist)

```
set<int> Adj[MAXN]; vector<int> Circuit;
void Hierholzer() {
    int v = 0;          // find node with odd degree, else start with node 0
    for (int u=0; u < N && v == 0; u++)
        if (Adj[u].size() & 1)
            v = u;          // node with odd degree
    stack<int> Stack;
    Stack.push(v);
    while (!Stack.empty()) {
        if (!Adj[v].empty()) {          // follow edges until stuck
            Stack.push(v);
            int tmp = *Adj[v].begin();
            Adj[v].erase(tmp);          // remove edge, modifying graph
            Adj[tmp].erase(v);
            v = tmp;
        } else {          // got stuck: stack contains a circuit
            Circuit.push_back(v);      // append node at the end of circuit
            v = Stack.top();          // backtrack using stack, find larger circuit
            Stack.pop();
        }
    }
}
https://www-m9.ma.tum.de/graph-algorithms/hierholzer/index\_en.html
}
```