Topological Sorting under Regular Constraints

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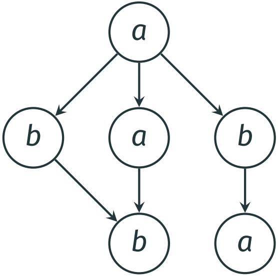

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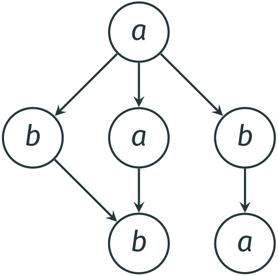

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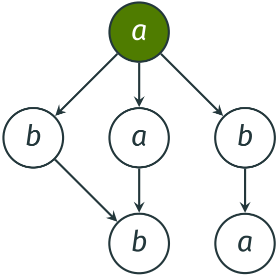

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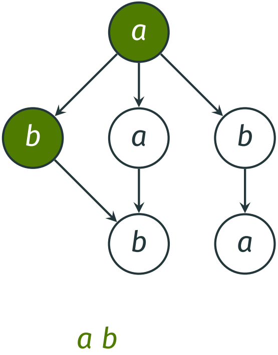

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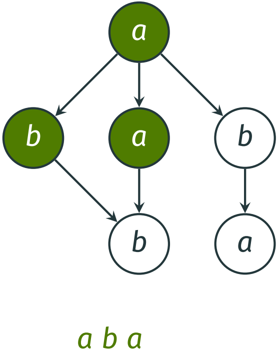

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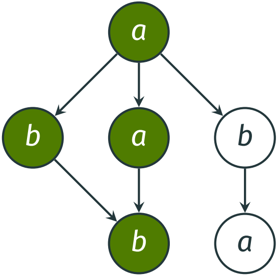

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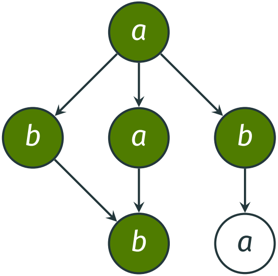

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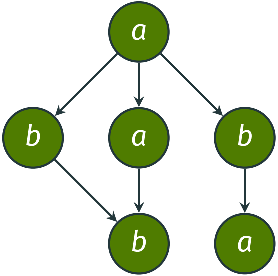

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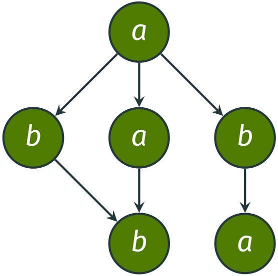

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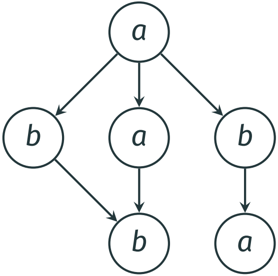

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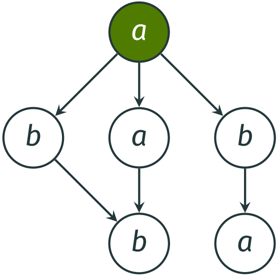

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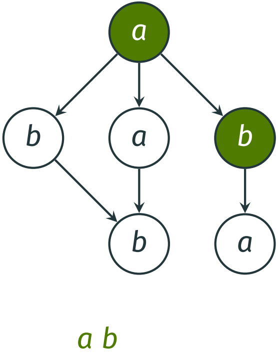

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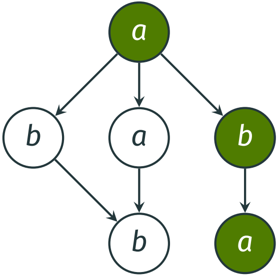

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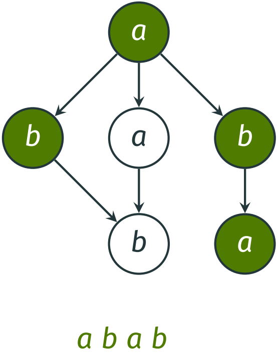

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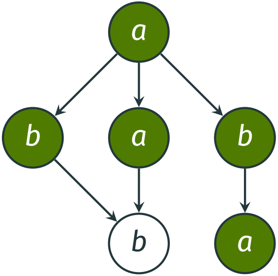

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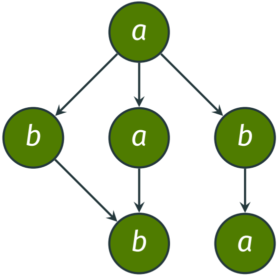

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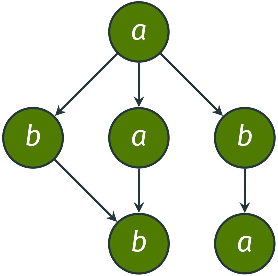

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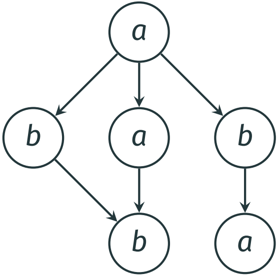

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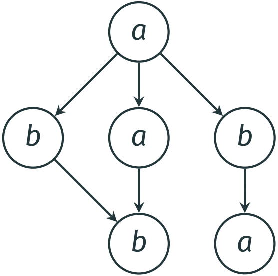
- **Question:** when is this problem tractable?



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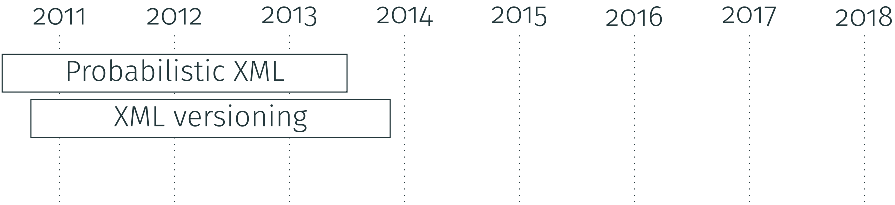

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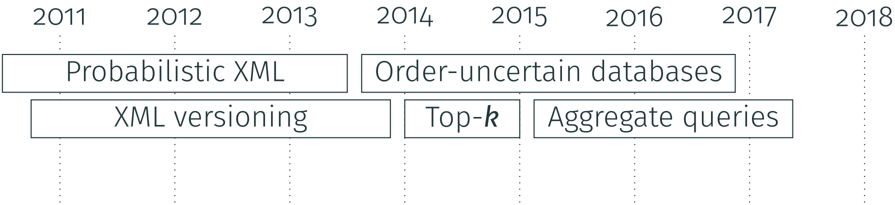

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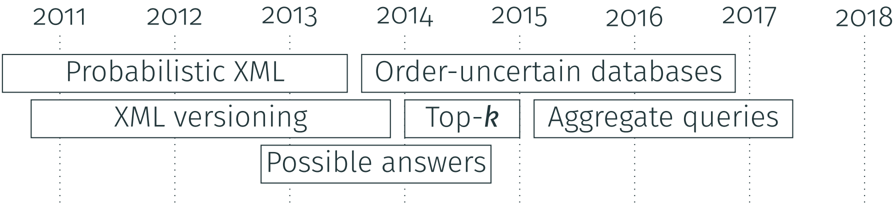

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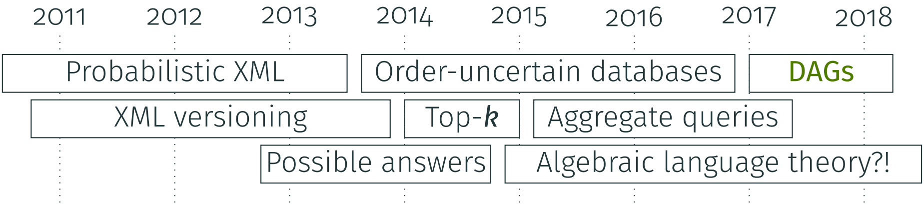

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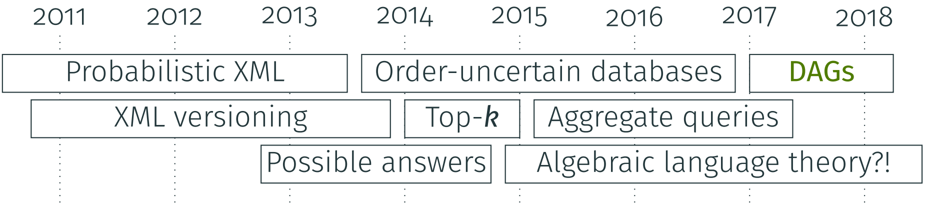

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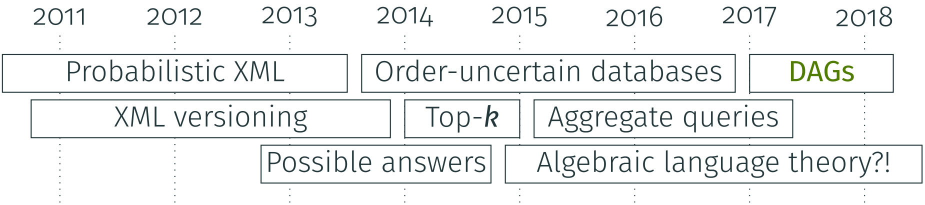

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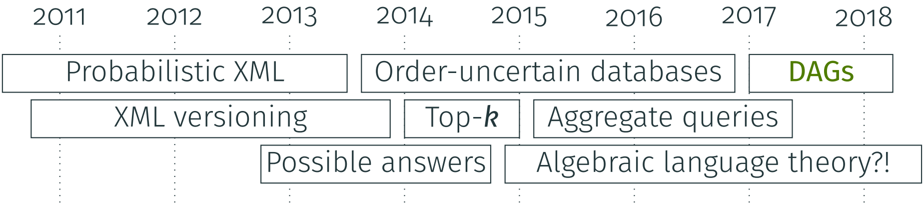

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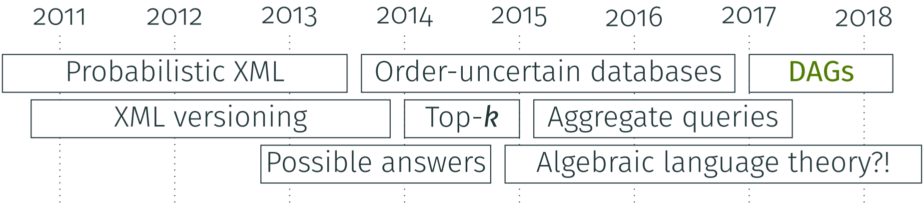

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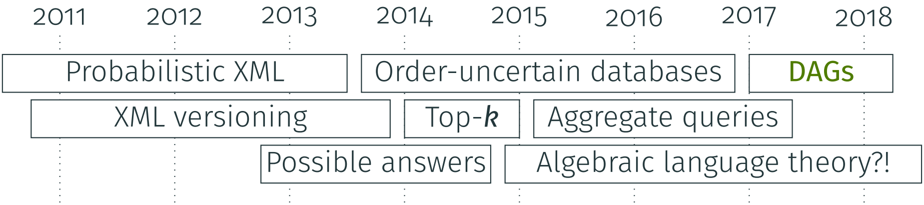

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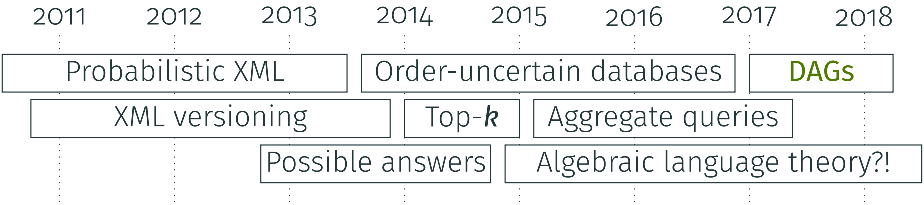

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$\rightarrow$ **Natural** problem and apparently **nothing** was known about it!

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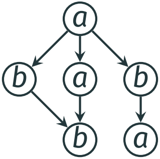

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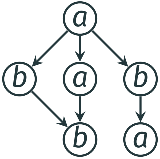

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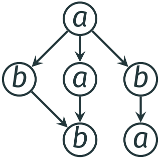

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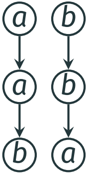

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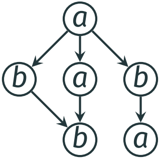

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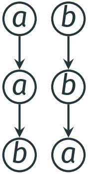

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$\rightarrow$ Question: What is the complexity of $\mathsf{CTS}(L)$ and $\mathsf{CSh}(L)$, depending on the fixed language L ?

Dichotomy

*For every **regular language L**, exactly one of the following holds:*

- *L has [some nice property] and* CTS(L) *is **in NL***
- *L has [some nasty property] and* CTS(L) *is **NP-hard***

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→ Very mysterious landscape! (to us)

Hardness Results

Existing Hardness Result

JOURNAL OF COMPUTER AND SYSTEM SCIENCES **28**, 345–358 (1984)

On the Complexity of Iterated Shuffle*

MANFRED K. WARMUTH[†] AND DAVID HAUSSLER[‡]

Department of Computer Science,
University of Colorado, Boulder, Colorado 80309

It is demonstrated that the following problems are *NP* complete:

(1) Given words w and $w_1, w_2, \ldots, w_n$, is w in the shuffle of $w_1, w_2, \ldots, w_n$?

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$\rightarrow$ Does not directly apply for us, because we **fix** the target language

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- We can reduce their problem to CSh for the language $(aA + bB)^*$
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Tractability Results

Tractability for Monomials

- **Monomial:** language of the form $A_1^* \, a_1 \, A_2^* \, a_2 \, \cdots \, A_n^* \, a_n \, A_{n+1}^*$ where $a_1, \ldots, a_n \in \Sigma$ and $A_1, \ldots, A_{n+1} \subseteq \Sigma$
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- Check that the other vertices **can fit** in the A_i^* (uses NL = co-NL)

The Algebraic Approach

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Remark: For the language $L = b\Sigma^* + aa\Sigma^* + (ab)^*$

- CTS(L) is NP-hard because $(ab)^{-1}L = (ab)^*$
- CSh(L) is in NL: trivial if there is more than one word

Tractability Based on Width

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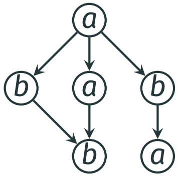

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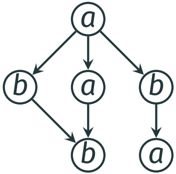

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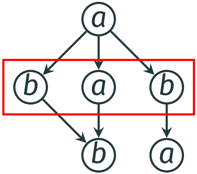

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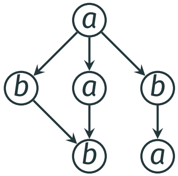

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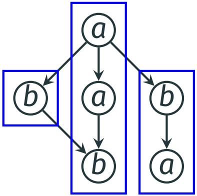

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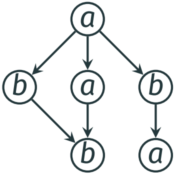

→ These results are making an additional assumption, but...

Tractability Based on Width (2)

- Fix $\Sigma = \{a, b\}$, take any regular language L and constant $k \in \mathbb{N}$, we know that CTS is in NL for $L + \Sigma^*(a^k + b^k)\Sigma^*$

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  → Unknown for $L + \Sigma^* a^k \Sigma^*$ with arbitrary L and $k > 2$! ¯_(ツ)_/¯

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  - What about similar languages like $(aa + bb + ab)^*$? ¯_(ツ)_/¯
- $(caa)^* d (cbb)^* d \Sigma^* + \Sigma^* cc \Sigma^*$ is in NL for CSh but NP-hard for CTS
  - Tractability argument: another ad hoc greedy algorithm
  - Hardness argument: from k-clique encoded to a bipartite graph

Conclusion

Summary and Future Work

Language	**CSh** (shuffle)	**CTS** (top. sort)
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Monomials… Groups, di…		in NL ¯_(ツ)_/¯
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$L + \Sigma^*(a^k$… $(ab)^* + \Sigma^*$… $L + \Sigma^* a^k \Sigma$…		in NL in NL ¯_(ツ)_/¯
$(aa+bb)^*$ $(aa+b)^*$ $(a^k+b)^*$		NP-hard ¯_(ツ)_/¯ ¯_(ツ)_/¯
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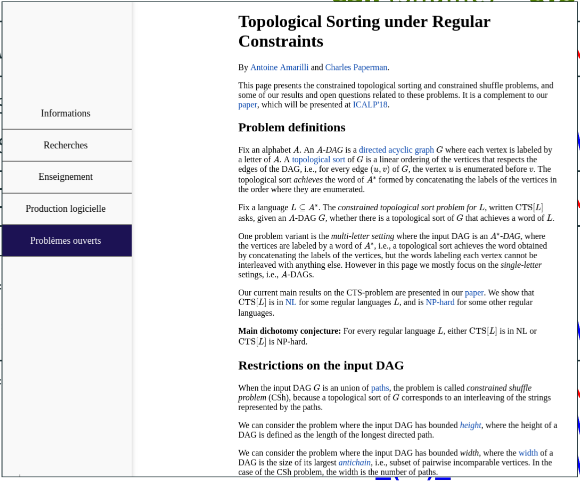

Informations

Recherches

Enseignement

Production logicielle

Problèmes ouverts

Topological Sorting under Regular Constraints

By Antoine Amarilli and Charles Paperman.

This page presents the constrained topological sorting and constrained shuffle problems, and some of our results and open questions related to these problems. It is a complement to our paper, which will be presented at ICALP'18.

Problem definitions

Fix an alphabet A. An *A-DAG* is a directed acyclic graph G where each vertex is labeled by a letter of A. A topological sort of G is a linear ordering of the vertices that respects the edges of the DAG, i.e., for every edge (u, v) of G, the vertex u is enumerated before v. The topological sort *achieves* the word of A^* formed by concatenating the labels of the vertices in the order where they are enumerated.

Fix a language $L \subseteq A^*$. The *constrained topological sort problem for L*, written CTS[L] asks, given an A-DAG G, whether there is a topological sort of G that achieves a word of L.

One problem variant is the *multi-letter setting* where the input DAG is an A^*-*DAG*, where the vertices are labeled by a word of A^*, i.e., a topological sort achieves the word obtained by concatenating the labels of the vertices, but the words labeling each vertex cannot be interleaved with anything else. However in this page we mostly focus on the *single-letter* setings, i.e., A-DAGs.

Our current main results on the CTS-problem are presented in our paper. We show that CTS[L] is in NL for some regular languages L, and is NP-hard for some other regular languages.

Main dichotomy conjecture: For every regular language L, either CTS[L] is in NL or CTS[L] is NP-hard.

Restrictions on the input DAG

When the input DAG G is an union of paths, the problem is called *constrained shuffle problem* (CSh), because a topological sort of G corresponds to an interleaving of the strings represented by the paths.

We can consider the problem where the input DAG has bounded *height*, where the height of a DAG is defined as the length of the longest directed path.

We can consider the problem where the input DAG has bounded *width*, where the width of a DAG is the size of its largest *antichain*, i.e., subset of pairwise incomparable vertices. In the case of the CSh problem, the width is the number of paths.

Summary and Future Work

Language		CTS (top. sort)
$(ab)^*$, u^* with		NP-hard
Monomials A_1^* Groups, distric		in NL ¯_(ツ)_/¯
$b\Sigma^* + aa\Sigma^* +$		NP-hard
$L + \Sigma^*(a^k + b^k$ $(ab)^* + \Sigma^* a^2\Sigma$ $L + \Sigma^* a^k \Sigma^*$		in NL in NL ¯_(ツ)_/¯
$(aa + bb)^*$, $(a$ $(aa + b)^*$ $(a^k + b)^*$		NP-hard ¯_(ツ)_/¯ ¯_(ツ)_/¯
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$(ab)^* + \Sigma^* a^2 \Sigma^*$	in NL	in NL
$L + \Sigma^* a^k \Sigma^*$	¯_(ツ)_/¯	¯_(ツ)_/¯
$(aa + bb)^*$, $(ab + a)^*$	NP-hard	NP-hard
$(aa + b)^*$	in NL	¯_(ツ)_/¯
$(a^k + b)^*$	¯_(ツ)_/¯	¯_(ツ)_/¯
Essentially all other languages...	¯_(ツ)_/¯	¯_(ツ)_/¯

Thanks for your attention!

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