Conditional logics: from models to automated reasoning

Marianna Girlando

University of Birmingham

Based on joint work with: Björn Lellmann, Nicola Olivetti, Stefano Pesce and Gian Luca Pozzato

DIG team Seminar
Paris, 25 March 2022

Outline

- Conditional logics
- Models
- Proof theory and automated reasoning

Conditional logics

Conditionals in natural language

If A *then* B

Conditionals in natural language

If A then B

- *If I hadn't overslept, then I would have caught the train.*

Conditionals in natural language

If A then B

- *If I hadn't overslept, then I would have caught the train.*
- *If Tux is a bird then it can fly.*

Conditionals in natural language

If A then B

- *If I hadn't overslept, then I would have caught the train.*
- *If Tux is a bird then it can fly.*
  But if Tux is a bird and a penguin, then it can't fly.

Conditionals in natural language

If A then B

- *If I hadn't overslept, then I would have caught the train.*
- *If Tux is a bird then it can fly.*
  But if Tux is a bird and a penguin, then it can't fly.
   *Normally, birds can fly*.

Conditionals in natural language

If A *then* B

- *If I hadn't overslept, then I would have caught the train.*
- *If Tux is a bird then it can fly.*
  But if Tux is a bird and a penguin, then it can't fly.
   *Normally, birds can fly*.
- *If Alice saw a lunar eclipse, then she would no longer believe that Earth is flat.*
- ...

Paradoxes of material implication

Paradoxes of material implication

- *If I hadn't overslept, then I would have caught the train.*

Paradoxes of material implication

- *If I hadn't overslept, then I would have caught the train.*

A	B	$A \to B$
T	T	T
T	F	F
F	T	T
F	F	T

Paradoxes of material implication

- *If I hadn't overslept, then I would have caught the train.*

A	B	$A \to B$
T	T	T
T	F	F
F	T	T
F	F	T

- *If Tux is a bird then it can fly.*
  But if Tux is a bird and a penguin, then it can't fly.

Paradoxes of material implication

- *If I hadn't overslept, then I would have caught the train.*

A	B	$A \to B$
T	T	T
T	F	F
F	T	T
F	F	T

- *If Tux is a bird then it can fly.*
  But if Tux is a bird and a penguin, then it can't fly.

$$\textit{Monotonicity} \quad (A \to B) \to ((A \wedge C) \to B)$$

Conditionals in a modal framework

Conditionals in a modal framework

$$A, B \ ::= \ p \mid \bot \mid A \to B$$

Conditionals in a modal framework

$$A, B ::= p \mid \bot \mid A \to B$$

$$\neg A := A \to \bot \qquad A \wedge B := \neg(A \to \neg B) \qquad A \vee B := \neg A \to B$$

Conditionals in a modal framework

$$A, B \;::=\; p \mid \bot \mid A \to B \mid \Box A$$

$$\neg A := A \to \bot \qquad A \wedge B := \neg(A \to \neg B) \qquad A \vee B := \neg A \to B$$

Conditionals in a modal framework

$$A, B ::= p \mid \bot \mid A \to B \mid \Box A$$

$$\neg A := A \to \bot \qquad A \wedge B := \neg(A \to \neg B) \qquad A \vee B := \neg A \to B$$

1960-70: Stalnaker, Lewis, Nute, Chellas, Burgess ...

Conditionals in a modal framework

$$A, B \ ::= \ p \mid \bot \mid A \to B \mid \Box A$$

$$\neg A := A \to \bot \qquad A \wedge B := \neg(A \to \neg B) \qquad A \vee B := \neg A \to B$$

1960-70: Stalnaker, Lewis, Nute, Chellas, Burgess ...

$$A, B \ ::= \ p \mid \bot \mid A \to B$$

Conditionals in a modal framework

$$A, B \ ::= \ p \mid \bot \mid A \to B \mid \Box A$$

$$\neg A := A \to \bot \qquad A \wedge B := \neg(A \to \neg B) \qquad A \vee B := \neg A \to B$$

1960-70: Stalnaker, Lewis, Nute, Chellas, Burgess . . .

$$A, B \ ::= \ p \mid \bot \mid A \to B \mid A > B$$

Conditionals in a modal framework

$$A, B \ ::= \ p \mid \bot \mid A \to B \mid \Box A$$

$$\neg A := A \to \bot \qquad A \wedge B := \neg(A \to \neg B) \qquad A \vee B := \neg A \to B$$

1960-70: Stalnaker, Lewis, Nute, Chellas, Burgess . . .

$$A, B \ ::= \ p \mid \bot \mid A \to B \mid A > B$$

$$\Box A := \neg A > \bot$$

Conditionals in a modal framework

$$A, B ::= p \mid \bot \mid A \to B \mid \Box A$$

$$\neg A := A \to \bot \qquad A \wedge B := \neg(A \to \neg B) \qquad A \vee B := \neg A \to B$$

1960-70: Stalnaker, Lewis, Nute, Chellas, Burgess . . .

$$A, B ::= p \mid \bot \mid A \to B \mid A \preccurlyeq B$$

$$\Box A := \neg A > \bot$$

"A is at least as plausible as B"

$$A > B := (\bot \preccurlyeq A) \vee \neg((A \wedge \neg B) \preccurlyeq (A \vee B))$$

Conditionals in a modal framework

$$A, B ::= p \mid \bot \mid A \to B \mid \Box A$$

$$\neg A := A \to \bot \qquad A \wedge B := \neg(A \to \neg B) \qquad A \vee B := \neg A \to B$$

1960-70: Stalnaker, Lewis, Nute, Chellas, Burgess . . .

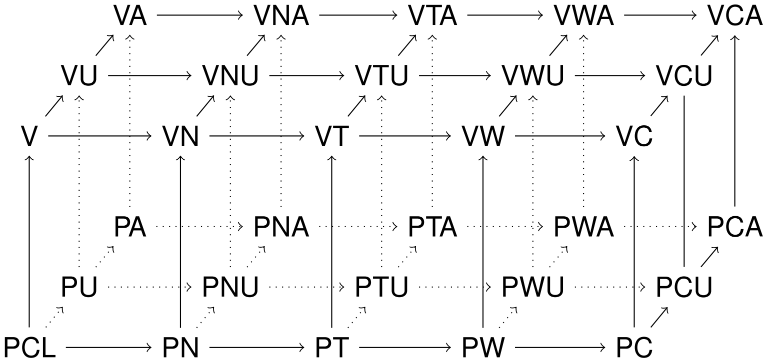
$$A, B ::= p \mid \bot \mid A \to B \mid A \preccurlyeq B$$

$$\Box A := \neg A > \bot \qquad \Box A := \bot \preccurlyeq \neg A$$

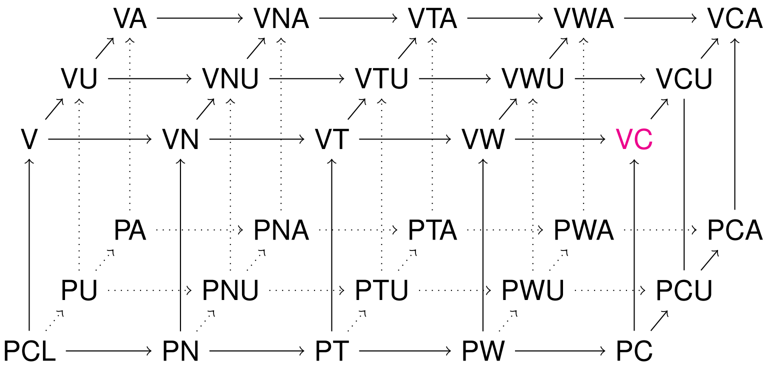
"A is at least as plausible as B"

$$A > B := (\bot \preccurlyeq A) \vee \neg((A \wedge \neg B) \preccurlyeq (A \vee B))$$

Conditional logics

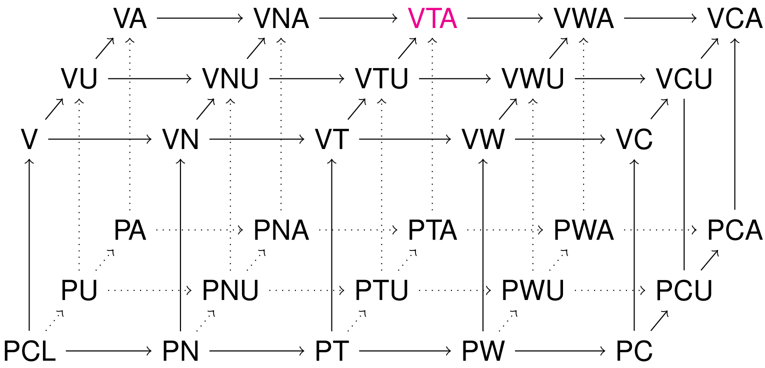


Conditional logics



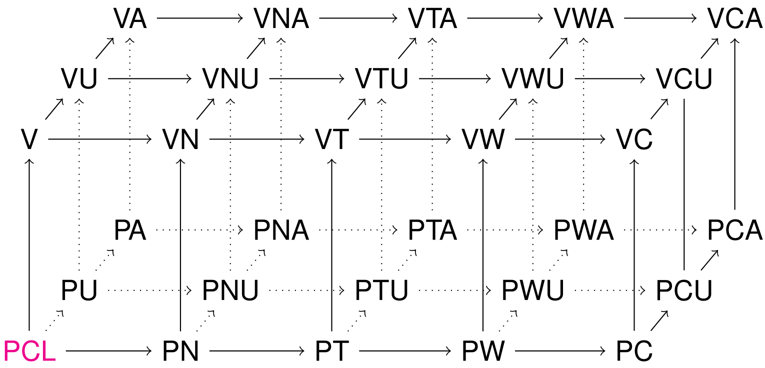
- Counterfactuals [Lewis,1973]

Conditional logics



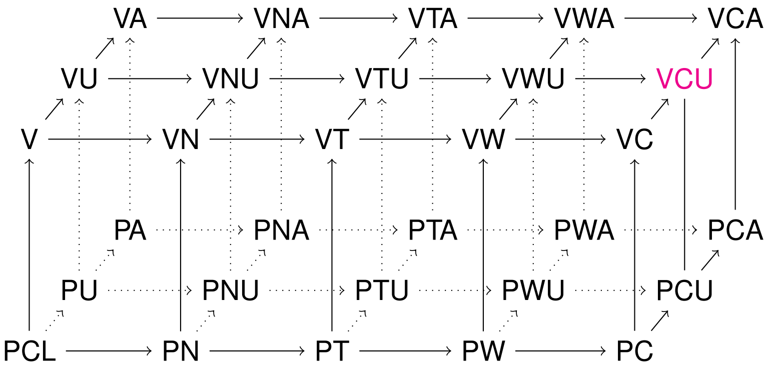
- Counterfactuals [Lewis,1973]
- Conditional belief of agents [Baltag and Smets,2006]

Conditional logics



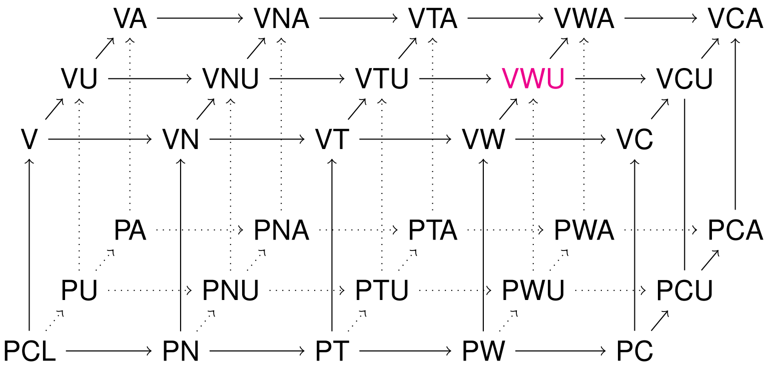
- Counterfactuals [Lewis,1973]
- Conditional belief of agents [Baltag and Smets,2006]
- Prototypical properties [KLM, 1990]

Conditional logics



- Counterfactuals [Lewis,1973]
- Conditional belief of agents [Baltag and Smets,2006]
- Prototypical properties [KLM, 1990]
- Knowledge bases update [Grahne, 1998]

Conditional logics



- Counterfactuals [Lewis,1973]
- Conditional belief of agents [Baltag and Smets,2006]
- Prototypical properties [KLM, 1990]
- Knowledge bases update [Grahne, 1998]
- Knowledge bases with preferences [Sheremet et al., 2007]

Knowledge bases update [Katsuno and Mendelzon, 1991]

Knowledge bases update [Katsuno and Mendelzon, 1991]

B, C: formulas

Knowledge bases update [Katsuno and Mendelzon, 1991]

B, C: formulas

K: knowledge base (conjunction of formulas)

Knowledge bases update [Katsuno and Mendelzon, 1991]

B, C: formulas

K: knowledge base (conjunction of formulas)

$K \circ B$: operation of updating K with B

Knowledge bases update [Katsuno and Mendelzon, 1991]

B, C: formulas

K: knowledge base (conjunction of formulas)

$K \circ B$: operation of updating K with B

$$\left\{ \mathit{Bird} \rightarrow \mathit{Flies}, \mathit{Tux} \ \rightarrow \ \mathit{Bird} \right\}$$

Knowledge bases update [Katsuno and Mendelzon, 1991]

B, C: formulas

K: knowledge base (conjunction of formulas)

$K \circ B$: operation of updating K with B

$$\left\{ \mathit{Bird} \rightarrow \mathit{Flies},\ \mathit{Tux} \rightarrow \mathit{Bird} \right\} \circ \mathit{Tux} \rightarrow \mathit{notFlies}$$

Knowledge bases update [Katsuno and Mendelzon, 1991]

B, C: formulas

K: knowledge base (conjunction of formulas)

$K \circ B$: operation of updating K with B

$$\left\{ Bird \rightarrow Flies,\ Tux \rightarrow Bird \right\} \circ Tux \rightarrow notFlies$$

$$K \circ A \rightarrow A$$

$$\text{if } K \rightarrow A \text{ then } K \circ A \equiv K$$

$$\vdots$$

Knowledge bases update [Katsuno and Mendelzon, 1991]

B, C: formulas

K: knowledge base (conjunction of formulas)

$K \circ B$: operation of updating K with B

$$\left\{ Bird \rightarrow Flies,\ Tux \ \rightarrow \ Bird \right\} \circ Tux \rightarrow notFlies$$

$$K \circ A \rightarrow A$$

$$\text{if } K \rightarrow A \text{ then } K \circ A \equiv K$$

$$\vdots$$

[Grahne, 1998] Conditional logic VCU + update operator

Knowledge bases update [Katsuno and Mendelzon, 1991]

B, C: formulas

K: knowledge base (conjunction of formulas)

$K \circ B$: operation of updating K with B

$$\left\{ Bird \rightarrow Flies,\ Tux \rightarrow Bird \right\} \circ Tux \rightarrow notFlies$$

$$K \circ A \rightarrow A$$

$$\text{if } K \rightarrow A \text{ then } K \circ A \equiv K$$

$$\vdots$$

[Grahne, 1998] Conditional logic VCU + update operator

$$(K \circ B) \rightarrow C \text{ holds} \quad \textit{iff} \quad K \rightarrow (B > C) \text{ holds}$$

Knowledge bases with preferences

Knowledge bases with preferences

Logic of Comparative Concepts Similarity [Sheremet et al., 2007]

Knowledge bases with preferences

Logic of Comparative Concepts Similarity [Sheremet et al., 2007]

Description logic + preference operator $\leftleftarrows$

Knowledge bases with preferences

Logic of Comparative Concepts Similarity [Sheremet et al., 2007]

Description logic + preference operator $\leftleftarrows$

$$Blueish \equiv (\,Blue \leftleftarrows Red\,)$$

Knowledge bases with preferences

Logic of Comparative Concepts Similarity [Sheremet et al., 2007]

Description logic + preference operator $\Leftarrow$

$$Blueish \equiv (Blue \Leftarrow Red)$$

$$Turquoise \sqsubseteq (Blue \Leftarrow Red)$$

Knowledge bases with preferences

Logic of Comparative Concepts Similarity [Sheremet et al., 2007]

Description logic + preference operator $\leftleftarrows$

$$Blueish \equiv (Blue \leftleftarrows Red)$$

$$Turquoise \sqsubseteq (Blue \leftleftarrows Red)$$

Under the *limit assumption*, $\leftleftarrows$ and $>$ of VWU have the same expressive power.

Models for conditional logics

Neighbourhood models for VC

$\mathcal{M} = \langle W, N, [\![\cdot]\!] \rangle$

Neighbourhood models for VC

$\mathcal{M} = \langle W, N, [\![\cdot]\!] \rangle$

y

x

z

k

Neighbourhood models for VC

$\mathcal{M} = \langle W, N, [\![\cdot]\!] \rangle \quad N : W \to \mathcal{P}(\mathcal{P}(W))$ s.t. $\emptyset \notin N(x)$

y

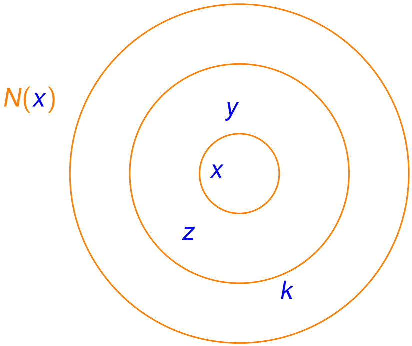
x

z

k

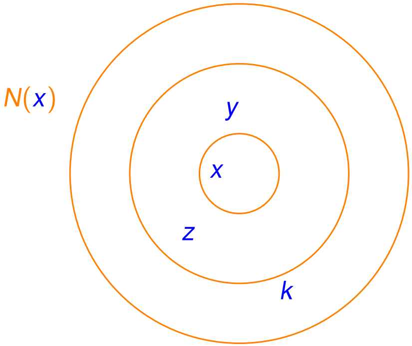
Neighbourhood models for VC

$\mathcal{M} = \langle W, N, [\![\cdot]\!] \rangle \quad N : W \to \mathcal{P}(\mathcal{P}(W))$ s.t. $\emptyset \notin N(x)$



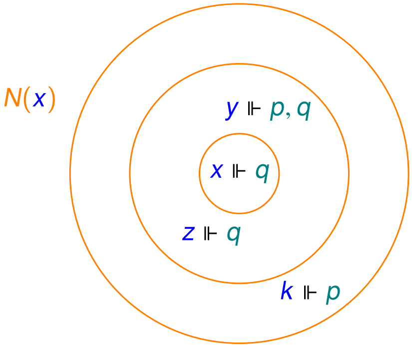
Neighbourhood models for VC

$\mathcal{M} = \langle W, N, [\![\cdot]\!] \rangle \quad N : W \to \mathcal{P}(\mathcal{P}(W))$ s.t. $\emptyset \notin N(x) \quad [\![\cdot]\!] : Atm \to \mathcal{P}(W)$



Neighbourhood models for VC

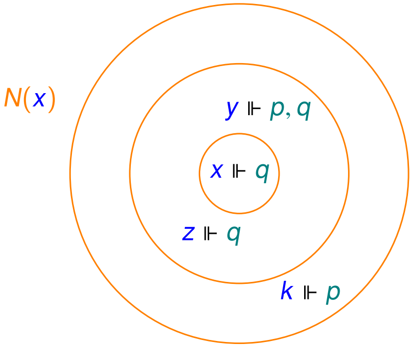
$\mathcal{M} = \langle W, N, [\![\cdot]\!] \rangle \quad N : W \to \mathcal{P}(\mathcal{P}(W))$ s.t. $\emptyset \notin N(x) \quad [\![\cdot]\!] : Atm \to \mathcal{P}(W)$



Neighbourhood models for VC

$\mathcal{M} = \langle W, N, [\![\cdot]\!] \rangle \quad N : W \to \mathcal{P}(\mathcal{P}(W))$ s.t. $\emptyset \notin N(x) \quad [\![\cdot]\!] : Atm \to \mathcal{P}(W)$

Nesting for all $\alpha, \beta \in N(x)$, $\alpha \subseteq \beta$ or $\beta \subseteq \alpha$

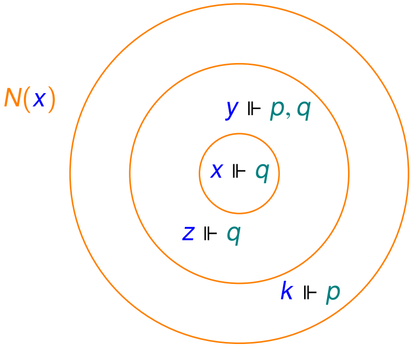


Neighbourhood models for VC

$\mathcal{M} = \langle W, N, [\![\cdot]\!] \rangle$ $N : W \to \mathcal{P}(\mathcal{P}(W))$ s.t. $\emptyset \notin N(x)$ $[\![\cdot]\!] : Atm \to \mathcal{P}(W)$

Nesting   for all $\alpha, \beta \in N(x)$, $\alpha \subseteq \beta$ or $\beta \subseteq \alpha$

Centering   for all $\alpha \in N(x)$, $x \in \alpha$ and $\{x\} \in N(x)$

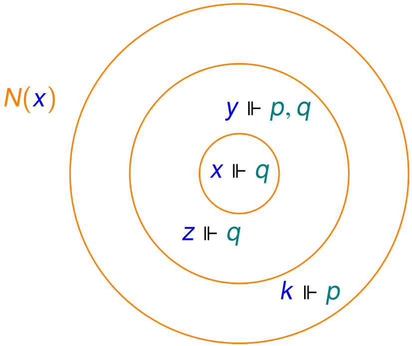


Neighbourhood models for VC

$\mathcal{M} = \langle W, N, [\![\cdot]\!] \rangle \quad N : W \to \mathcal{P}(\mathcal{P}(W))$ s.t. $\emptyset \notin N(x) \quad [\![\cdot]\!] : Atm \to \mathcal{P}(W)$

Nesting for all $\alpha, \beta \in N(x)$, $\alpha \subseteq \beta$ or $\beta \subseteq \alpha$

Centering for all $\alpha \in N(x)$, $x \in \alpha$ and $\{x\} \in N(x)$



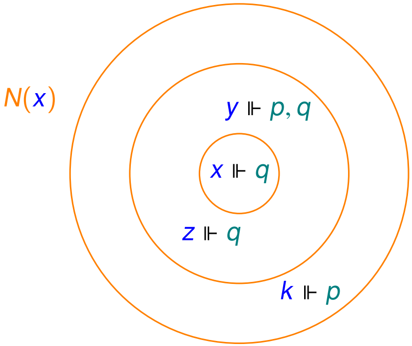
$x \Vdash q \preccurlyeq p$ *iff* for all $\alpha \in N(x)$, if $\alpha \Vdash^{\exists} p$ then $\alpha \Vdash^{\exists} q$

Neighbourhood models for VC

$\mathcal{M} = \langle W, N, [\![\cdot]\!]\rangle$ $\quad N : W \to \mathcal{P}(\mathcal{P}(W))$ s.t. $\emptyset \notin N(x)$ $\quad [\![\cdot]\!] : Atm \to \mathcal{P}(W)$

Nesting for all $\alpha, \beta \in N(x)$, $\alpha \subseteq \beta$ or $\beta \subseteq \alpha$

Centering for all $\alpha \in N(x)$, $x \in \alpha$ and $\{x\} \in N(x)$



$x \Vdash q \preccurlyeq p$ *iff* for all $\alpha \in N(x)$, if $\alpha \Vdash^{\exists} p$ then $\alpha \Vdash^{\exists} q$

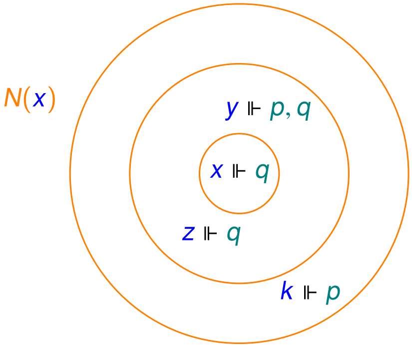
$\alpha \Vdash^{\forall} A \equiv \forall y \in \alpha, y \Vdash A$ $\qquad$ $\alpha \Vdash^{\exists} A \equiv \exists y \in \alpha$ s. t. $y \Vdash A$

Neighbourhood models for VC

$\mathcal{M} = \langle W, N, [\![\cdot]\!] \rangle$ $\quad N : W \to \mathcal{P}(\mathcal{P}(W))$ s.t. $\emptyset \notin N(x)$ $\quad [\![\cdot]\!] : Atm \to \mathcal{P}(W)$

Nesting for all $\alpha, \beta \in N(x)$, $\alpha \subseteq \beta$ or $\beta \subseteq \alpha$

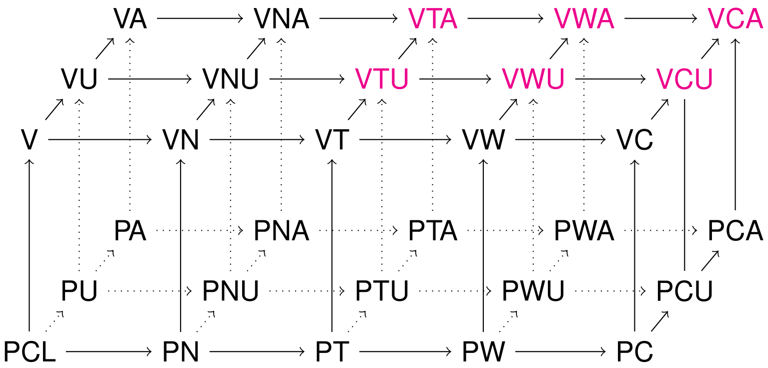
Centering for all $\alpha \in N(x)$, $x \in \alpha$ and $\{x\} \in N(x)$



$x \Vdash p > q$ *iff* if there is $\alpha \in N(x)$ s.t. $\alpha \Vdash^{\exists} p$,
there is $\beta \in N(x)$ s.t. $\beta \Vdash^{\exists} p$ and $\beta \Vdash^{\forall} p \to q$

$\alpha \Vdash^{\forall} A \equiv \forall y \in \alpha, y \Vdash A$ $\qquad$ $\alpha \Vdash^{\exists} A \equiv \exists y \in \alpha$ s. t. $y \Vdash A$

Conditional logics



Nes for all $\alpha, \beta \in N(x)$, $\alpha \subseteq \beta$ or $\beta \subseteq \alpha$

N $N(x) \neq \emptyset$

T there is $\alpha \in N(x)$ s.t. $x \in \alpha$

W $N(x) \neq \emptyset$ and for all $\alpha \in N(x)$, $x \in \alpha$

C $\{x\} \in N(x)$ and for all $\alpha \in N(x)$, $x \in \alpha$

U for all x, y, $\bigcup N(x) = \bigcup N(y)$

A for all x, y, $N(x) = N(y)$

Proof theory and automated reasoning

Proof systems for propositional logic

Sequent calculus [Gentzen, 1933-34]

Proof systems for propositional logic

Sequent calculus [Gentzen, 1933-34]

Γ, Δ multisets of formulas

$$\Gamma \Rightarrow \Delta \rightsquigarrow \bigwedge \Gamma \rightarrow \bigvee \Delta$$

Proof systems for propositional logic

Sequent calculus [Gentzen, 1933-34]

Γ, Δ multisets of formulas

$$\Gamma \Rightarrow \Delta \rightsquigarrow \bigwedge \Gamma \rightarrow \bigvee \Delta$$

$$\text{init}\ \frac{}{p, \Gamma \Rightarrow \Delta, p}$$

$$\to_R \frac{A, \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \to B} \qquad \vee_L \frac{A, \Gamma \Rightarrow \Delta \quad B, \Gamma \Rightarrow \Delta}{A \vee B, \Gamma \Rightarrow \Delta} \qquad \neg_L \frac{\Gamma \Rightarrow \Delta, A}{\neg A, \Gamma \Rightarrow \Delta}$$

Proof systems for propositional logic

Sequent calculus [Gentzen, 1933-34]

Γ, Δ multisets of formulas

$$\Gamma \Rightarrow \Delta \rightsquigarrow \bigwedge \Gamma \to \bigvee \Delta$$

$$\text{init}\ \frac{}{p, \Gamma \Rightarrow \Delta, p}$$

$$\to_R \frac{A, \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \to B} \qquad \vee_L \frac{A, \Gamma \Rightarrow \Delta \quad B, \Gamma \Rightarrow \Delta}{A \vee B, \Gamma \Rightarrow \Delta} \qquad \neg_L \frac{\Gamma \Rightarrow \Delta, A}{\neg A, \Gamma \Rightarrow \Delta}$$

$$\to_R \cfrac{\to_R \cfrac{\vee_L \cfrac{\neg_L \cfrac{\text{init}\ \cfrac{}{A \Rightarrow A}}{\neg A, A \Rightarrow} \qquad \text{init}\ \cfrac{}{B \Rightarrow B}}{\neg A \vee B, A \Rightarrow B}}{\neg A \vee B \Rightarrow A \to B}}{\Rightarrow (\neg A \vee B) \to (A \to B)}$$

Proof systems for modal logics

Proof systems for modal logics

$$\Box \frac{\Sigma \Rightarrow A}{\Box\Sigma, \Gamma \Rightarrow \Delta, \Box A}$$

$\Box\Sigma = \Box B_1, \ldots, \Box B_k$, for $0 \leq k$

Proof systems for modal logics

$$\Box\,\frac{\Sigma \Rightarrow A}{\Box\Sigma, \Gamma \Rightarrow \Delta, \Box A}$$

$\Box\Sigma = \Box B_1, \ldots, \Box B_k$, for $0 \leq k$

Problem for some systems of modal logics (S5), it is not possible to define cut-free Gentzen-style sequent calculi

$$\text{cut}\,\frac{\Gamma \Rightarrow \Delta, A \quad A, \Gamma' \Rightarrow \Delta'}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'}$$

Proof systems for modal logics

$$\Box \frac{\Sigma \Rightarrow A}{\Box\Sigma, \Gamma \Rightarrow \Delta, \Box A}$$

$\Box\Sigma = \Box B_1, \ldots, \Box B_k$, for $0 \leq k$

Problem for some systems of modal logics (S5), it is not possible to define cut-free Gentzen-style sequent calculi

$$\text{cut}\ \frac{\Gamma \Rightarrow \Delta, A \quad A, \Gamma' \Rightarrow \Delta'}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'}$$

Solution enrich the *structure* of sequents

Proof systems for modal logics

$$\square\ \frac{\Sigma \Rightarrow A}{\square\Sigma, \Gamma \Rightarrow \Delta, \square A}$$

$\square\Sigma = \square B_1, \ldots, \square B_k$, for $0 \leq k$

Problem for some systems of modal logics (S5), it is not possible to define cut-free Gentzen-style sequent calculi

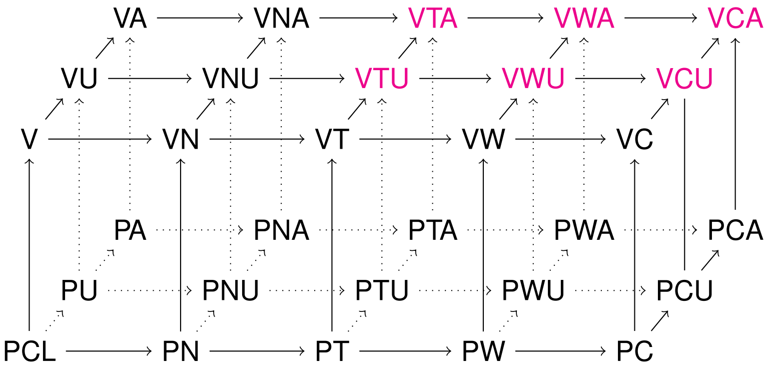
$$\text{cut}\ \frac{\Gamma \Rightarrow \Delta, A \quad A, \Gamma' \Rightarrow \Delta'}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'}$$

Solution enrich the *structure* of sequents

Hypersequent calculus [Avron, 1996]

$$\Gamma_1 \Rightarrow \Delta_1 \mid \cdots \mid \Gamma_n \Rightarrow \Delta_n \rightsquigarrow \square(\bigwedge \Gamma_1 \rightarrow \Delta_1) \vee \ldots \vee \square(\bigwedge \Gamma_n \rightarrow \Delta_n)$$

Proof systems for conditional logics



Nes for all $\alpha, \beta \in N(x)$, $\alpha \subseteq \beta$ or $\beta \subseteq \alpha$

T there is $\alpha \in N(x)$ s.t. $x \in \alpha$

U for all x, y, $\bigcup N(x) = \bigcup N(y)$

Hypersequents with blocks [G, Lellmann, Olivetti, Pozzato, 2017]

Hypersequents with blocks [G, Lellmann, Olivetti, Pozzato, 2017]

$$[C_1, \ldots, C_n \lhd B] \rightsquigarrow (C_1 \preccurlyeq B) \vee \cdots \vee (C_n \preccurlyeq B)$$

Hypersequents with blocks [G, Lellmann, Olivetti, Pozzato, 2017]

$$[C_1, \ldots, C_n \lhd B] \rightsquigarrow (C_1 \preccurlyeq B) \vee \cdots \vee (C_n \preccurlyeq B)$$

$$\langle C_1, \ldots, C_m \rangle \rightsquigarrow \neg(\bot \preccurlyeq (C_1 \vee \ldots \vee C_m))$$

Hypersequents with blocks [G, Lellmann, Olivetti, Pozzato, 2017]

$$[C_1, \ldots, C_n \lhd B] \rightsquigarrow (C_1 \preccurlyeq B) \vee \cdots \vee (C_n \preccurlyeq B)$$

$$\langle C_1, \ldots, C_m \rangle \rightsquigarrow \neg(\bot \preccurlyeq (C_1 \vee \ldots \vee C_m))$$

$$\Gamma \Rightarrow \Delta, [\Sigma_1 \lhd C_1], \ldots, [\Sigma_n \lhd C_n], \langle \Theta_1 \rangle, \ldots, \langle \Theta_m \rangle$$

Hypersequents with blocks [G, Lellmann, Olivetti, Pozzato, 2017]

$$[C_1, \ldots, C_n \lhd B] \rightsquigarrow (C_1 \preccurlyeq B) \vee \cdots \vee (C_n \preccurlyeq B)$$

$$\langle C_1, \ldots, C_m \rangle \rightsquigarrow \neg(\bot \preccurlyeq (C_1 \vee \ldots \vee C_m))$$

$$\Gamma \Rightarrow \Delta, [\Sigma_1 \lhd C_1], \ldots, [\Sigma_n \lhd C_n], \langle \Theta_1 \rangle, \ldots, \langle \Theta_m \rangle$$

$$\rightsquigarrow \bigwedge \Gamma \rightarrow \bigvee \Delta \vee \Big(\bigvee_{i=1}^{n} \bigvee_{B \in \Sigma_i} B \preccurlyeq C_i \Big) \vee \Big(\bigvee_{j=1}^{m} \neg\big(\bot \preccurlyeq \bigvee \Theta_j\big) \Big)$$

Hypersequents with blocks [G, Lellmann, Olivetti, Pozzato, 2017]

$$[C_1, \ldots, C_n \lhd B] \rightsquigarrow (C_1 \preccurlyeq B) \vee \cdots \vee (C_n \preccurlyeq B)$$

$$\langle C_1, \ldots, C_m \rangle \rightsquigarrow \neg(\bot \preccurlyeq (C_1 \vee \ldots \vee C_m))$$

$$\Gamma \Rightarrow \Delta, [\Sigma_1 \lhd C_1], \ldots, [\Sigma_n \lhd C_n], \langle \Theta_1 \rangle, \ldots, \langle \Theta_m \rangle$$

$$\rightsquigarrow \bigwedge \Gamma \rightarrow \bigvee \Delta \vee \Big(\bigvee_{i=1}^{n} \bigvee_{B \in \Sigma_i} B \preccurlyeq C_i \Big) \vee \Big(\bigvee_{j=1}^{m} \neg\big(\bot \preccurlyeq \bigvee \Theta_j\big) \Big)$$

$$\Gamma_1 \Rightarrow \Delta_1 \mid \cdots \mid \Gamma_n \Rightarrow \Delta_n$$
$$\rightsquigarrow \Box(\bigwedge \Gamma_1 \rightarrow \bigvee \Delta_1) \vee \cdots \vee \Box(\bigwedge \Gamma_n \rightarrow \bigvee \Delta_n)$$

$$\Box A \equiv \bot \preccurlyeq \neg A$$

Example

Example

$$\preccurlyeq_{\mathrm{R}}\ \frac{\mathcal{G} \mid \Gamma \Rightarrow \Delta, [A \lhd B]}{\mathcal{G} \mid \Gamma \Rightarrow \Delta, A \preccurlyeq B}$$

Example

$$\preccurlyeq_R \frac{\mathcal{G} \mid \Gamma \Rightarrow \Delta, [A \lhd B]}{\mathcal{G} \mid \Gamma \Rightarrow \Delta, A \preccurlyeq B} \qquad \text{jump} \frac{\mathcal{G} \mid \Gamma \Rightarrow \Delta \mid C \Rightarrow \Sigma}{\mathcal{G} \mid \Gamma \Rightarrow \Delta, [\Sigma \lhd C]}$$

Example

$$\preccurlyeq_R \frac{\mathcal{G} \mid \Gamma \Rightarrow \Delta, [A \lhd B]}{\mathcal{G} \mid \Gamma \Rightarrow \Delta, A \preccurlyeq B} \quad \text{jump} \frac{\mathcal{G} \mid \Gamma \Rightarrow \Delta \mid C \Rightarrow \Sigma}{\mathcal{G} \mid \Gamma \Rightarrow \Delta, [\Sigma \lhd C]}$$

$$\text{jump}_U \frac{\mathcal{G} \mid \Gamma \Rightarrow \Delta \mid \Sigma \Rightarrow \Pi, \Theta}{\mathcal{G} \mid \Gamma \Rightarrow \Delta, \langle \Theta \rangle \mid \Sigma \Rightarrow \Pi}$$

Example

$$\frac{\mathcal{G} \mid \Gamma \Rightarrow \Delta, [A \lhd B]}{\mathcal{G} \mid \Gamma \Rightarrow \Delta, A \preccurlyeq B} \preccurlyeq_R \qquad \frac{\mathcal{G} \mid \Gamma \Rightarrow \Delta \mid C \Rightarrow \Sigma}{\mathcal{G} \mid \Gamma \Rightarrow \Delta, [\Sigma \lhd C]} \text{jump}$$

$$\frac{\mathcal{G} \mid \Gamma \Rightarrow \Delta \mid \Sigma \Rightarrow \Pi, \Theta}{\mathcal{G} \mid \Gamma \Rightarrow \Delta, \langle \Theta \rangle \mid \Sigma \Rightarrow \Pi} \text{jump}_U$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\perp \Rightarrow \perp \qquad \dfrac{\Rightarrow [\perp \lhd \perp \preccurlyeq A], [\perp \lhd A] \mid \perp \preccurlyeq A \Rightarrow \perp, \langle \perp, A \rangle \mid A \Rightarrow \perp, A}{\Rightarrow [\perp \lhd \perp \preccurlyeq A], [\perp \lhd A] \mid \perp \preccurlyeq A \Rightarrow \perp, \langle \perp, A \rangle \mid A \Rightarrow \perp} \text{jump}_U}{\Rightarrow [\perp \lhd \perp \preccurlyeq A], [\perp \lhd A] \mid \perp \preccurlyeq A \Rightarrow \perp, \langle \perp \rangle \mid A \Rightarrow \perp} \text{T}}{\Rightarrow [\perp \lhd \perp \preccurlyeq A], [\perp \lhd A] \mid \perp \preccurlyeq A \Rightarrow \perp \mid A \Rightarrow \perp} \text{int}}{\Rightarrow [\perp \lhd \perp \preccurlyeq A], [\perp \lhd A]} \text{jump(2x)}}{\Rightarrow \perp \preccurlyeq (\perp \preccurlyeq A), \perp \preccurlyeq A} \preccurlyeq_R\text{(2x)}}{\neg(\perp \preccurlyeq A) \Rightarrow \perp \preccurlyeq (\perp \preccurlyeq A)} \neg_L}{\Rightarrow \neg(\perp \preccurlyeq A) \to (\perp \preccurlyeq (\perp \preccurlyeq A))} \to_R$$

Example

$$
\leqslant_R\,\frac{\mathcal{G} \mid \Gamma \Rightarrow \Delta, [A \lhd B]}{\mathcal{G} \mid \Gamma \Rightarrow \Delta, A \preccurlyeq B}
\qquad
\text{jump}\,\frac{\mathcal{G} \mid \Gamma \Rightarrow \Delta \mid C \Rightarrow \Sigma}{\mathcal{G} \mid \Gamma \Rightarrow \Delta, [\Sigma \lhd C]}
\qquad
\text{int}\,\frac{\mathcal{G} \mid \Gamma \Rightarrow \Delta, \langle \bot \rangle}{\mathcal{G} \mid \Gamma \Rightarrow \Delta}
$$

$$
\text{T}\,\frac{\mathcal{G} \mid \Gamma \Rightarrow \Delta \mid A \Rightarrow \Theta \qquad \mathcal{G} \mid \Gamma \Rightarrow \Delta, \langle \Theta, B \rangle}{\mathcal{G} \mid A \preccurlyeq B, \Gamma \Rightarrow \Delta, \langle \Theta \rangle}
\qquad
\text{jump}_U\,\frac{\mathcal{G} \mid \Gamma \Rightarrow \Delta \mid \Sigma \Rightarrow \Pi, \Theta}{\mathcal{G} \mid \Gamma \Rightarrow \Delta, \langle \Theta \rangle \mid \Sigma \Rightarrow \Pi}
$$

$$
\text{T}\,\dfrac{\bot \Rightarrow \bot \qquad \text{jump}_U\,\dfrac{\Rightarrow [\bot \lhd \bot \preccurlyeq A], [\bot \lhd A] \mid \bot \preccurlyeq A \Rightarrow \bot, \langle \bot, A \rangle \mid A \Rightarrow \bot, A}{\Rightarrow [\bot \lhd \bot \preccurlyeq A], [\bot \lhd A] \mid \bot \preccurlyeq A \Rightarrow \bot, \langle \bot, A \rangle \mid A \Rightarrow \bot}}{\text{int}\,\dfrac{\Rightarrow [\bot \lhd \bot \preccurlyeq A], [\bot \lhd A] \mid \bot \preccurlyeq A \Rightarrow \bot, \langle \bot \rangle \mid A \Rightarrow \bot}{\text{jump(2x)}\,\dfrac{\Rightarrow [\bot \lhd \bot \preccurlyeq A], [\bot \lhd A] \mid \bot \preccurlyeq A \Rightarrow \bot \mid A \Rightarrow \bot}{\leqslant_R\text{(2x)}\,\dfrac{\Rightarrow [\bot \lhd \bot \preccurlyeq A], [\bot \lhd A]}{\neg_L\,\dfrac{\Rightarrow \bot \preccurlyeq (\bot \preccurlyeq A), \bot \preccurlyeq A}{\to_R\,\dfrac{\neg(\bot \preccurlyeq A) \Rightarrow \bot \preccurlyeq (\bot \preccurlyeq A)}{\Rightarrow \neg(\bot \preccurlyeq A) \to (\bot \preccurlyeq (\bot \preccurlyeq A))}}}}}}
$$

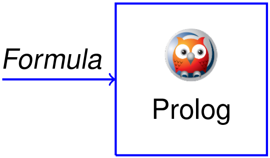
What about implementing hypersequents?

Theorem prover [G, Lellmann, Olivetti, Pesce, Pozzato, 2022]

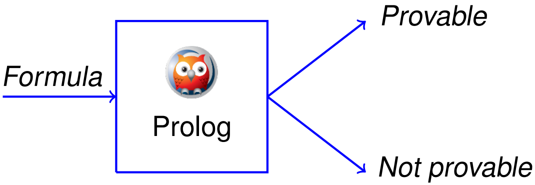
Theorem prover [G, Lellmann, Olivetti, Pesce, Pozzato, 2022]

Prolog

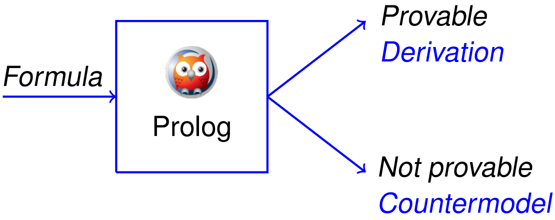
Theorem prover [G, Lellmann, Olivetti, Pesce, Pozzato, 2022]



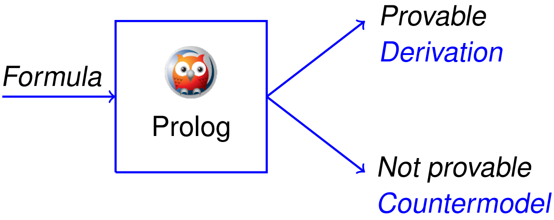
Theorem prover [G, Lellmann, Olivetti, Pesce, Pozzato, 2022]



Theorem prover [G, Lellmann, Olivetti, Pesce, Pozzato, 2022]



Theorem prover [G, Lellmann, Olivetti, Pesce, Pozzato, 2022]

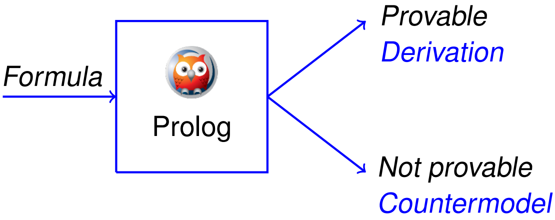


TuCLEVER

Total reflexivity and Uniformity

Conditional LEwis logics theorem proVER

Theorem prover [G, Lellmann, Olivetti, Pesce, Pozzato, 2022]



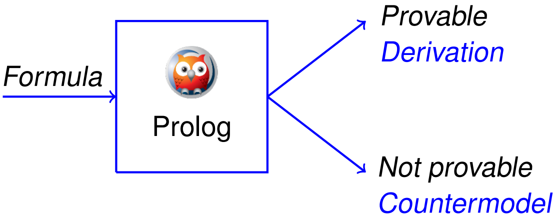
TuCLEVER

Total reflexivity and Uniformity

Conditional LEwis logics theorem proVER

- Proof search for VTU, VWU, VCU, VTA, VWA and VCA

Theorem prover [G, Lellmann, Olivetti, Pesce, Pozzato, 2022]



TuCLEVER

Total reflexivity and Uniformity

Conditional LEwis logics theorem proVER

- Proof search for VTU, VWU, VCU, VTA, VWA and VCA
- Countermodel construction for VTU, VWU and VCU

How TuCLEVER works?

How TuCLEVER works?

- Implements proof search

How TuCLEVER works?

- Implements proof search

```
prove(Hypersequent,tree(condR,Hypersequent,[Gamma,Delta],no,SubTree1,no)) :-
        select([Gamma,Delta],Hypersequent,Remainder),
        member(A < B, Delta),
        \+findBlock(Delta,[[A],B]),!,
        prove([[Gamma,[[[A],B]|Delta]]|Remainder],SubTree1).
```

$$\preccurlyeq_R \frac{\mathcal{G} \mid \Gamma \Rightarrow \Delta, [A \lhd B]}{\mathcal{G} \mid \Gamma \Rightarrow \Delta, A \preccurlyeq B}$$

How TuCLEVER works?

- Implements proof search

```
prove(Hypersequent,tree(condR,Hypersequent,[Gamma,Delta],no,SubTree1,no)) :-
        select([Gamma,Delta],Hypersequent,Remainder),
        member(A < B, Delta),
        \+findBlock(Delta,[[A],B]),!,
        prove([[Gamma,[[[A],B]|Delta]]|Remainder],SubTree1).
```

$$\preccurlyeq_R \frac{\mathcal{G} \mid \Gamma \Rightarrow \Delta, [A \lhd B]}{\mathcal{G} \mid \Gamma \Rightarrow \Delta, A \preccurlyeq B}$$

- Implements termination strategy

How TuCLEVER works?

- Implements proof search

```
prove(Hypersequent,tree(condR,Hypersequent,[Gamma,Delta],no,SubTree1,no)) :-
        select([Gamma,Delta],Hypersequent,Remainder),
        member(A < B, Delta),
        \+findBlock(Delta,[[A],B]),!,
        prove([[Gamma,[[[A],B]|Delta]]|Remainder],SubTree1).
```

$$\preccurlyeq_R \frac{\mathcal{G} \mid \Gamma \Rightarrow \Delta, [A \lhd B]}{\mathcal{G} \mid \Gamma \Rightarrow \Delta, A \preccurlyeq B}$$

- Implements termination strategy

$\Rightarrow a \preccurlyeq (a \rightarrow \neg b), b \preccurlyeq (\neg a \rightarrow b), [a, b \lhd a \rightarrow \neg b], [b \lhd \neg a \rightarrow b], \langle \bot \rangle, \bot \mid$
$a \rightarrow \neg b \Rightarrow a, b, b, \langle \bot \rangle, \bot \mid \neg a \rightarrow b, a \Rightarrow b, \langle \bot \rangle, \bot$

How TuCLEVER works?

- Implements proof search

```
prove(Hypersequent,tree(condR,Hypersequent,[Gamma,Delta],no,SubTree1,no)) :-
        select([Gamma,Delta],Hypersequent,Remainder),
        member(A < B, Delta),
        \+findBlock(Delta,[[A],B]),!,
        prove([[Gamma,[[[A],B]|Delta]]|Remainder],SubTree1).
```

$$\preccurlyeq_R \frac{\mathcal{G} \mid \Gamma \Rightarrow \Delta, [A \lhd B]}{\mathcal{G} \mid \Gamma \Rightarrow \Delta, A \preccurlyeq B}$$

- Implements termination strategy

  $\Rightarrow a \preccurlyeq (a \rightarrow \neg b), b \preccurlyeq (\neg a \rightarrow b), [a, b \lhd a \rightarrow \neg b], [b \lhd \neg a \rightarrow b], \langle\bot\rangle, \bot \mid$

  $a \rightarrow \neg b \Rightarrow a, b, b, \langle\bot\rangle, \bot \mid \neg a \rightarrow b, a \Rightarrow b, \langle\bot\rangle, \bot$
- Implements countermodel construction

How TuCLEVER works?

- Implements proof search

```
prove(Hypersequent,tree(condR,Hypersequent,[Gamma,Delta],no,SubTree1,no)) :-
        select([Gamma,Delta],Hypersequent,Remainder),
        member(A < B, Delta),
        \+findBlock(Delta,[[A],B]),!,
        prove([[Gamma,[[[A],B]|Delta]]|Remainder],SubTree1).
```

$$\preccurlyeq_R \frac{\mathcal{G} \mid \Gamma \Rightarrow \Delta, [A \lhd B]}{\mathcal{G} \mid \Gamma \Rightarrow \Delta, A \preccurlyeq B}$$

- Implements termination strategy

  $\Rightarrow a \preccurlyeq (a \to \neg b), b \preccurlyeq (\neg a \to b), [a, b \lhd a \to \neg b], [b \lhd \neg a \to b], \langle \bot \rangle, \bot \mid$

  $a \to \neg b \Rightarrow a, b, b, \langle \bot \rangle, \bot \mid \neg a \to b, a \Rightarrow b, \langle \bot \rangle, \bot$

- Implements countermodel construction

  $\langle W_{\mathcal{H}}, N_{\mathcal{H}}, [\![\ ]\!]_{\mathcal{H}} \rangle$

  $W_{\mathcal{H}} = \{1, 2, 3\}$

  $N_{\mathcal{H}}(1) = \{\{2\}, \{2, 3\}, W_{\mathcal{H}}\}$ $N_{\mathcal{H}}(2) = N_{\mathcal{H}}(3) = W_{\mathcal{H}}$

  $[\![a]\!]_{\mathcal{H}} = \{3\}$ $[\![b]\!]_{\mathcal{H}} = \emptyset$

Let's try it!

http://193.51.60.97:8000/tuclever/

Let's try it!

http://193.51.60.97:8000/tuclever/

- Is $(a \preccurlyeq b) \vee (b \preccurlyeq a)$ derivable in VTU?

Let's try it!

http://193.51.60.97:8000/tuclever/

- Is $(a \preccurlyeq b) \vee (b \preccurlyeq a)$ derivable in VTU?
- Is $a \preccurlyeq (\neg a \vee b) \vee b \preccurlyeq (a \vee b)$ derivable in VTU?

Let's try it!

http://193.51.60.97:8000/tuclever/

- Is $(a \preccurlyeq b) \vee (b \preccurlyeq a)$ derivable in VTU? (Yes)
- Is $a \preccurlyeq (\neg a \vee b) \vee b \preccurlyeq (a \vee b)$ derivable in VTU? (No)

Conclusions

Conclusions

This talk:

Conclusions

This talk:

- Proof systems for conditional logics with nesting, total reflexivity and uniformity

Conclusions

This talk:

- Proof systems for conditional logics with nesting, total reflexivity and uniformity
- TuCLEVER

Conclusions

This talk:

- Proof systems for conditional logics with nesting, total reflexivity and uniformity
- TuCLEVER

Future work:

Conclusions

This talk:

- Proof systems for conditional logics with nesting, total reflexivity and uniformity
- TuCLEVER

Future work:

- Explore the proof theory of the lower layer of the lattice

Conclusions

This talk:

- Proof systems for conditional logics with nesting, total reflexivity and uniformity
- TuCLEVER

Future work:

- Explore the proof theory of the lower layer of the lattice
- Explore the proof theory of logics with the $\preccurlyeq$ operator

Conclusions

This talk:

- Proof systems for conditional logics with nesting, total reflexivity and uniformity
- TuCLEVER

Future work:

- Explore the proof theory of the lower layer of the lattice
- Explore the proof theory of logics with the $\preccurlyeq$ operator
- Explore further applications within knowledge bases . . .

Thank you!

Questions?

Axiom systems (I)

PCL

Axiomatization classical propositional logic plus

(RCEA) $$\frac{(A > C) \leftrightarrow (B > C)}{A \leftrightarrow B}$$

(RCK) $$\frac{(C > A) \rightarrow (C > B)}{A \rightarrow B}$$

(R-And) $(A > B) \wedge (A > C) \rightarrow (A > (B \wedge C))$

(ID) $A > A$

(CM) $(A > B) \wedge (A > C) \rightarrow ((A \wedge B) > C)$

(RT) $(A > B) \wedge ((A \wedge B) > C) \rightarrow (A > C)$

(OR) $(A > C) \wedge (B > C) \rightarrow ((A \vee B) > C)$

V

Axiomatization of PCL plus

(CV) $(A > C) \wedge \neg(A > \neg B) \rightarrow ((A \wedge B) > C)$

Axiom systems (II)

Axioms for extensions

(N)	$\neg(\top > \bot)$	*Normality*
(T)	$A \rightarrow \neg(A > \bot)$	*Total reflexivity*
(W)	$(A > B) \rightarrow (A \rightarrow B)$	*Weak centering*
(C)	$(A \wedge B) \rightarrow (A > B)$	*Strong centering*
(U_1)	$(\neg A > \bot) \rightarrow (\neg(\neg A > \bot) > \bot)$	*Uniformity (1)*
(U_2)	$\neg(A > \bot) \rightarrow ((A > \bot) > \bot)$	*Uniformity (2)*
(A_1)	$(A > B) \rightarrow (C > (A > B))$	*Absoluteness (1)*
(A_2)	$\neg(A > B) \rightarrow (C > \neg(A > B))$	*Absoluteness (2)*