Descriptive complexity for counting complexity classes

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Joint work with
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$$\begin{array}{ccc}
\mathrm{NP} & \equiv & \exists\mathsf{SO} \\
\textsc{co}\mathrm{NP} & \equiv & \forall\mathsf{SO} \\
\mathrm{P} & \equiv & \mathsf{LFP}_{\leq} \\
\mathrm{NL} & \equiv & \mathsf{TC}_{\leq} \\
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\mathrm{PSPACE} & \equiv & \mathsf{PFP}_{\leq} \\
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Many **applications** in diverse areas like:

1. Computational complexity and logics.
2. Database management systems.
3. Verification systems.

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One would like to study the **complexity** of problems like:

"How many valuations satisfies my boolean formula?"

"How many simple paths are connecting two vertices in my graph?"

...but computational complexity is not only about `true` or `false`

$$\#\mathrm{P}$$

$$\textsc{SpanP}$$

$$\mathrm{FP}$$

$$\#\mathrm{L}$$

$$\#\mathrm{PSPACE}$$

$$\vdots$$

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$$\text{Counting complexity classes} \begin{cases} \#\mathrm{P} \\ \textsc{SpanP} \\ \mathrm{FP} \\ \#\mathrm{L} \\ \#\mathrm{PSPACE} \\ \vdots \end{cases}$$

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How can we describe these **counting** classes with logic?

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   that have good **tractability** and **closure** properties.

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1. We show that QSO **captures** many counting complexity classes.
   - $\#\mathrm{P}$, $\textsc{SpanP}$, FP, $\#\mathrm{PSPACE}$, $\textsc{MinP}$, $\textsc{MaxP}$, $\ldots$
2. We use QSO to find classes below $\#\mathrm{P}$ that have good **tractability** and **closure** properties.
3. We show how to define **quantitative recursion** over QSO in order to capture classes below FP.

Outline

Quantitative second order logic

QSO vs counting complexity

Below and beyond

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QSO vs counting complexity

Below and beyond

Some notation and restrictions

Given a relational signature $\mathbf{R} = \{R_1, \ldots, R_k, <\}$,
we consider **finite ordered structures** over $\mathbf{R}$ of the form:

$$\mathfrak{A} = (A, R_1^{\mathfrak{A}}, \ldots, R_k^{\mathfrak{A}}, <^{\mathfrak{A}})$$

where A is the domain and $<^{\mathfrak{A}}$ is a **linear order** over A.

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We consider formulas of **Second Order logic** over $\mathbf{R}$ of the form:

$$\varphi \;:=\; \texttt{True} \mid x = y \mid R(\bar{u}) \mid X(\bar{v}) \mid \neg\varphi \mid (\varphi \vee \varphi) \mid \exists x.\, \varphi \mid \exists X.\, \varphi$$

where $R \in \mathbf{R}$ and x and X are a first and second order variable, respectively.

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The semantics of a second order formula is defined as usual.

Syntax of Quantitative Second Order logic

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Definition

A QSO-formula α over **R** is given by the following **syntax**:

$$\alpha \ := \ \varphi \in \mathsf{SO} \ \mid \ s \ \mid \ (\alpha + \alpha) \ \mid \ (\alpha \cdot \alpha) \ \mid \Sigma x.\,\alpha \ \mid \ \Pi x.\,\alpha \ \mid \ \Sigma X.\,\alpha \ \mid \ \Pi X.\,\alpha$$

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Example

Let $\mathbf{R} = \{E(\cdot,\cdot), <\}$ where E encodes an edge relation.

$$\alpha \;:=\; \Sigma x.\Sigma y.\Sigma z.\big(E(x,y) \wedge E(y,z) \wedge E(z,x) \wedge x < y \wedge y < z\big)$$

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Given a QSO-formula α, $\mathfrak{A} \in \textsc{Struct}(\mathbf{R})$ and a var. assignment $v : \mathbf{X} \to A$ we define the **semantics** $[\![\alpha]\!] : \textsc{Struct}(\mathbf{R}) \to \mathbb{N}$ recursively as follow:

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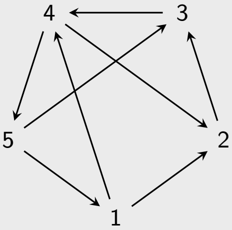
Semantics of Quantitative Second Order logic

Example (counting the triangles in a graph)

$\mathfrak{A}:$



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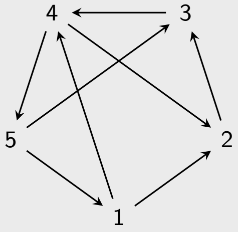

$$\text{triangle}(x,y,z) \;\coloneqq\; E(x,y) \wedge E(y,z) \wedge E(z,x) \wedge x < y \wedge y < z$$

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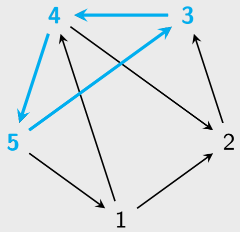

$$\text{triangle}(x, y, z) \;:=\; E(x, y) \wedge E(y, z) \wedge E(z, x) \wedge x < y \wedge y < z$$

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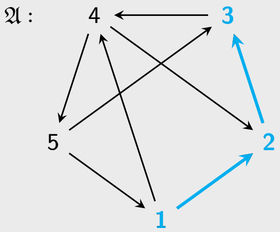

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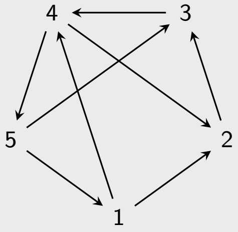

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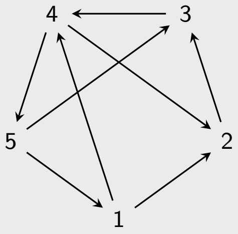

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$$\alpha \;:=\; \Sigma x.\,\Sigma y.\,\Sigma z.\ \text{triangle}(x,y,z)$$

$$[\![\alpha]\!](\mathfrak{A}) = 3$$

Semantics of Quantitative Second Order logic

Example (counting the number of cliques in a graph)

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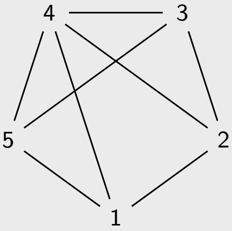

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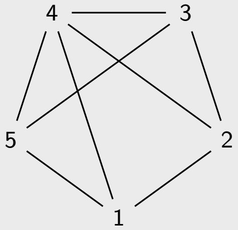

$$\text{clique}(X) \coloneqq \forall x.\, \forall y.\, (X(x) \wedge X(y) \wedge x \neq y) \rightarrow E(x, y)$$

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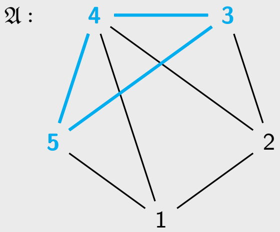

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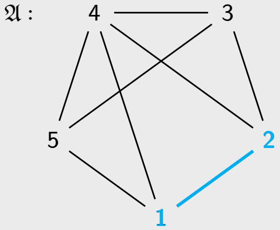

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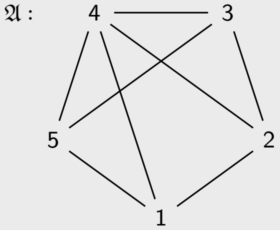

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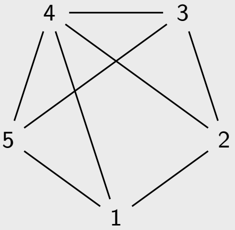

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$$[\![\alpha]\!](\mathfrak{A}) = 18$$

Subfragments and extentions of QSO

$$\alpha \;:=\; \varphi \in \mathsf{SO} \;\mid\; s \;\mid\; (\alpha + \alpha) \;\mid\; (\alpha \cdot \alpha) \;\mid \Sigma x.\,\alpha \;\mid\; \Pi x.\,\alpha \;\mid\; \Sigma X.\,\alpha \;\mid\; \Pi X.\,\alpha$$

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We can restrict or extend the language of φ:

QSO(FO) $:=$ φ is restricted to **FO logic**.

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We can restrict or extend the language of α:

$\mathsf{QFO(SO)}$ $:=$ α is restricted to **first order operators** (i.e. $s, +, \cdot, \Sigma x.\,, \Pi x.\,$).

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We can restrict or extend the language of α:

$\mathsf{QFO(SO)} \quad := \quad \alpha$ is restricted to **first order operators** (i.e. $s, +, \cdot, \Sigma x., \Pi x.$).
$\Sigma\mathsf{QSO(SO)} \quad := \quad \alpha$ is restricted to **sum operators** (i.e. $s, +, \Sigma x., \Sigma X.$)

Subfragments and extentions of QSO

$$\alpha \ := \ \varphi \in \mathsf{SO} \ \mid \ s \ \mid \ (\alpha + \alpha) \ \mid \ (\alpha \cdot \alpha) \ \mid \Sigma x.\, \alpha \ \mid \ \Pi x.\, \alpha \ \mid \ \Sigma X.\, \alpha \ \mid \ \Pi X.\, \alpha$$

$$\mathsf{QSO} \ = \ \underbrace{\mathsf{QSO}}_{\alpha}(\overbrace{\mathsf{SO}}^{\varphi})$$

We can restrict or extend the language of φ:

$$\begin{array}{rcl} \mathsf{QSO(FO)} & := & \varphi \text{ is restricted to } \textbf{FO logic}. \\ \mathsf{QSO(LFP)} & := & \varphi \text{ is restricted to } \textbf{LFP logic}. \end{array}$$

We can restrict or extend the language of α:

$$\begin{array}{rcl} \mathsf{QFO(SO)} & := & \alpha \text{ is restricted to } \textbf{first order operators} \text{ (i.e. } s, +, \cdot, \Sigma x., \Pi x.). \\ \Sigma\mathsf{QSO(SO)} & := & \alpha \text{ is restricted to } \textbf{sum operators} \text{ (i.e. } s, +, \Sigma x., \Sigma X.) \end{array}$$

Or both φ and α:

$$\begin{array}{rcl} \mathsf{QFO(LFP)} & = & \alpha \text{ is restricted to } \textbf{first order operators} \\ & & \text{and } \varphi \text{ is restricted to } \textbf{LFP logic}. \end{array}$$

Outline

Quantitative second order logic

QSO vs counting complexity

Below and beyond

Capturing a counting complexity class with QSO

- Recall that a counting complexity $\mathcal{C} \subseteq \{f : \Sigma^* \to \mathbb{N}\}$.

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Definition

Let $\mathcal{F}$ be a fragment or extension of QSO and $\mathcal{C}$ a counting complexity class. Then $\mathcal{F}$ **captures** $\mathcal{C}$ over ordered **R**-structures if:

1. for every $\alpha \in \mathcal{F}$, there exists $f \in \mathcal{C}$ such that $[\![\alpha]\!](\mathfrak{A}) = f(\mathrm{enc}(\mathcal{A}))$ for every $\mathfrak{A} \in \textsc{Struct}[\mathbf{R}]$.

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Capturing a counting complexity class with QSO

- Recall that a counting complexity $\mathcal{C} \subseteq \{f : \Sigma^* \to \mathbb{N}\}$.
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Definition

Let $\mathcal{F}$ be a fragment or extension of QSO and $\mathcal{C}$ a counting complexity class. Then $\mathcal{F}$ **captures** $\mathcal{C}$ over ordered **R**-structures if:

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$\mathcal{F}$ **captures** $\mathcal{C}$ over ordered structures if $\mathcal{F}$ captures $\mathcal{C}$ over ordered **R**-structures **for every signature R**.

What counting classes can be captured by QSO?

Counting complexity classes

$$\left\{\begin{array}{c} \#\mathrm{P} \\ \textsc{SpanP} \\ \mathrm{FP} \\ \#\mathrm{L} \\ \#\mathrm{PSPACE} \\ \vdots \end{array}\right.$$

What counting classes can be captured by QSO?

Counting complexity classes $\left\{ \begin{array}{c} \#\mathrm{P} \\ \textsc{SpanP} \\ \mathrm{FP} \\ \#\mathrm{L} \\ \#\mathrm{PSPACE} \\ \vdots \end{array} \right.$

We show that most of these classes
can be captured by subfragments or extensions of QSO

How to capture $\#\mathrm{P}$?

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Theorem

$\Sigma\mathrm{QSO(FO)}$ captures $\#\mathrm{P}$ over ordered structures.

How to capture $\textsc{SpanP}$?

$$\#\mathrm{P} \quad \equiv \quad \Sigma\mathsf{QSO}(\mathsf{FO})$$

How to capture SpanP?

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$f \in \textsc{SpanP}$ iff there exists an **NP machine** M with **output** such that $f(x) = \#\mathrm{outputs}_M(x)$ for all $x \in \Sigma^*$.

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Theorem

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Theorem

$\Sigma\mathsf{QSO}(\exists\mathsf{SO})$ captures SpanP over ordered structures.

$\#\mathrm{P}$ and SpanP were shown to be captured by a different framework of Saluja et al. and Compton et al.

How to capture FP?

$$
\begin{array}{rcl}
\#\mathrm{P} & \equiv & \Sigma\mathsf{QSO}(\mathsf{FO}) \\
\textsc{SpanP} & \equiv & \Sigma\mathsf{QSO}(\exists\mathsf{SO})
\end{array}
$$

How to capture FP?

$$\begin{array}{rcl} \#\mathrm{P} & \equiv & \Sigma\mathsf{QSO}(\mathsf{FO}) \\ \textsc{SpanP} & \equiv & \Sigma\mathsf{QSO}(\exists\mathsf{SO}) \end{array}$$

$f \in \mathrm{FP}$ iff there exists a **PTIME machine** M with output such that $f(x) = M(x)$ for all $x \in \Sigma^*$.

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Theorem

QFO(LFP) captures FP over ordered structures.

How to capture FPSPACE?

$$
\begin{array}{ccc}
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\textsc{SpanP} & \equiv & \Sigma\mathsf{QSO}(\exists\mathsf{SO}) \\
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\end{array}
$$

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Theorem

QSO(PFP) captures $\mathrm{FPSPACE}$ over ordered structures.

How to capture $\mathrm{FPSPACE}(\text{poly})$?

$$
\begin{array}{ccc}
\#\mathrm{P} & \equiv & \Sigma\mathsf{QSO}(\mathsf{FO}) \\
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\mathrm{FPSPACE} & \equiv & \mathsf{QSO}(\mathsf{PFP})
\end{array}
$$

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$f \in \mathrm{FPSPACE}(\text{poly})$ iff there exists a **PSPACE machine** M with output of polynomial size such that $f(x) = M(x)$ for all $x \in \Sigma^*$.

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Theorem

QFO(PFP) captures $\mathrm{FPSPACE}(\mathsf{poly})$ over ordered structures.

More classes?

$$\begin{array}{rcl}
\#\mathrm{P} & \equiv & \Sigma\mathsf{QSO}(\mathsf{FO}) \\
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\end{array}$$

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\textsc{GapP} & \equiv & \Sigma\mathsf{QSO}_{\mathbb{Z}}(\mathsf{FO}) \\
\textsc{MaxP} & \equiv & \mathsf{MaxQSO}(\mathsf{FO}) \\
\textsc{MinP} & \equiv & \mathsf{MinQSO}(\mathsf{FO})
\end{array}
$$

Outline

Use QSO to understand classes **below** $\#\mathrm{P}$

$$\#\mathrm{P} \quad \equiv \quad \Sigma\mathsf{QSO}(\mathsf{FO})$$

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We consider subfragments below FO:

$$\begin{aligned}
\Sigma_0 &= \{\ \theta \in \mathsf{FO} \mid \theta \text{ has no first-order quantifiers}\ \} \\
\Sigma_1 &= \{\ \varphi \in \mathsf{FO} \mid \varphi = \exists \bar{x}.\, \theta(\bar{x}) \ \wedge\ \theta \in \Sigma_0\ \} \\
\Pi_1 &= \{\ \varphi \in \mathsf{FO} \mid \varphi = \forall \bar{x}.\, \theta(\bar{x}) \ \wedge\ \theta \in \Sigma_0\ \} \\
\Sigma_2 &= \{\ \varphi \in \mathsf{FO} \mid \varphi = \exists \bar{x}.\, \forall \bar{y}.\, \theta(\bar{x}, \bar{y}) \ \wedge\ \theta \in \Sigma_0\ \} \\
\Pi_2 &= \{\ \varphi \in \mathsf{FO} \mid \varphi = \forall \bar{x}.\, \exists \bar{y}.\, \theta(\bar{x}, \bar{y}) \ \wedge\ \theta \in \Sigma_0\ \}
\end{aligned}$$

Use QSO to understand classes **below** $\#\mathrm{P}$

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Saluja et. al. counting classes below $\#\mathrm{P}$

$$\#\Sigma_0 \subsetneq \#\Sigma_1 \subsetneq \#\Pi_1 \subsetneq \#\Sigma_2 \subsetneq \#\Pi_2 = \#\mathsf{FO} \equiv \#\mathrm{P}$$

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Theorem (ΣQSO-hierarchy)

$$\begin{tikzcd}[column sep=small]
 & \#\Sigma_1 \\
\#\Sigma_0 \arrow[ur, phantom, sloped, "\subsetneq"] & \\
 & \Sigma\mathsf{QSO}(\Sigma_0) \arrow[ul, phantom, sloped, "\subsetneq"]
\end{tikzcd}$$

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 & \Sigma\mathsf{QSO}(\Sigma_0) \arrow[ur, phantom, sloped, "\subsetneq"] & & &
\end{tikzcd}$$

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 & \Sigma\mathrm{QSO}(\Sigma_0) \arrow[ur, phantom, sloped, "\subsetneq"] & & & & \\
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 & \Sigma\mathsf{QSO}(\Sigma_0) \arrow[ur, phantom, "\subsetneq"] & & & & &
\end{tikzcd}$$

$$\#\Sigma_2 = \Sigma\mathsf{QSO}(\Sigma_2) \subsetneq \#\Pi_2 = \Sigma\mathsf{QSO}(\Pi_2) \equiv \#\mathrm{P}$$

The class $\Sigma\mathsf{QSO}(\Sigma_1[\mathsf{FO}])$

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Consider the following fragment of FO:

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$$\Sigma_1[\mathrm{FO}] \quad = \quad \{\ \varphi \in \mathrm{FO} \ \mid\ \varphi = \exists \bar{x}.\, \theta(\bar{x}) \text{ and } \theta \text{ can contain atomic formulae of the form } u = v\,,\ X(\bar{u}) \textbf{ and } \boldsymbol{\varphi(\bar{u}) \in \mathrm{FO}}\ \}$$

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Theorem (good decision and closure properties)

The class $\Sigma\mathrm{QSO}(\Sigma_1[\mathrm{FO}])$ is closed under sum, multiplication and **subtraction by one**.

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Theorem (good decision and closure properties)

The class $\Sigma\mathrm{QSO}(\Sigma_1[\mathrm{FO}])$ is closed under sum, multiplication and **subtraction by one**. Moreover, $\Sigma\mathrm{QSO}(\Sigma_1[\mathrm{FO}]) \subseteq \mathrm{TotP}$

The class $\Sigma\text{QSO}(\Sigma_1[\text{FO}])$

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Theorem (good decision and closure properties)

The class $\Sigma\text{QSO}(\Sigma_1[\text{FO}])$ is closed under sum, multiplication and **subtraction by one**. Moreover, $\Sigma\text{QSO}(\Sigma_1[\text{FO}]) \subseteq \text{TotP}$ and every function in $\Sigma\text{QSO}(\Sigma_1[\text{FO}])$ has an FPRAS.

$\Sigma QSO(\Sigma_1[FO])$ is closed under subtraction by one

(proof sketch)

$\Sigma\mathsf{QSO}(\Sigma_1[\mathsf{FO}])$ is closed under subtraction by one

(proof sketch)

We focus on the case where $\alpha \in \Sigma\mathsf{QSO}(\Sigma_1[\mathsf{FO}])$ is of the form:

$$\alpha \quad = \quad \Sigma\bar{X}.\,\Sigma\bar{x}.\,\exists\bar{y}.\,\varphi(\bar{X},\bar{x},\bar{y})$$

$\Sigma\mathrm{QSO}(\Sigma_1[\mathrm{FO}])$ is closed under subtraction by one

(proof sketch)

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We construct a formula $\mathit{min}\text{-}\varphi^{\mathrm{FO}}(\bar{x})$ that identifies the **lexicographically minimal** assignment σ to $\bar{x}$ that satisfies $\exists\bar{X}.\, \exists\bar{y}.\, \varphi(\bar{X},\bar{x},\bar{y})$.

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In this setting, the existence of a **small witness** (in this case σ) is essential to have closure by subtraction by one.

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Thanks! Questions?