Topology of wireless networks

L. Decreusefond

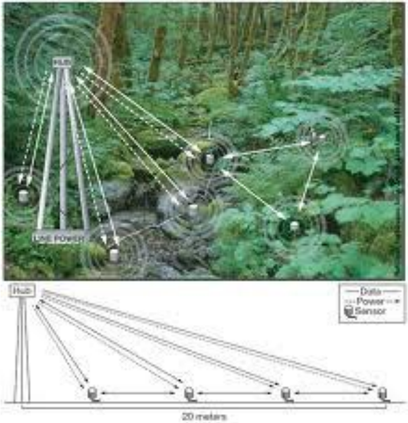
Institut
Mines-Telecom

Also starring (by chronological order of appearance)

P. Martins, E. Ferraz, F. Yan, A. Vergne, I. Flint, N.K. Le, A. Vasseur, (T. Bonis, B. Robert)

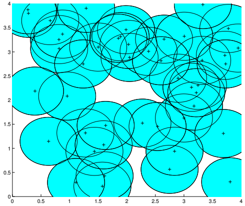
GANDI: Graphs ANalysis for Data and Information

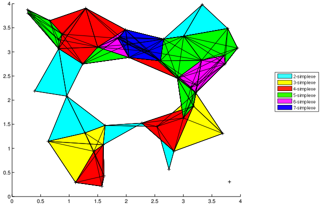
Applications : intelligent vehicle, agriculture, house, ...



Outline

Coverage

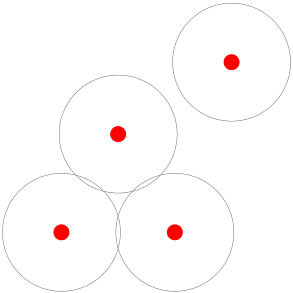



2-simplexe
3-simplexe
4-simplexe
5-simplexe
6-simplexe
7-simplexe

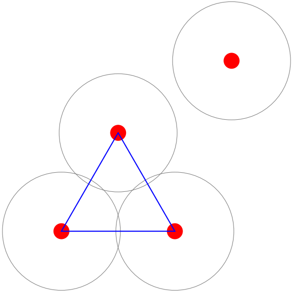
Mathematical framework

- Geometry leads to a combinatorial object
- Combinatorial object is equipped with a Linear algebra structure
- Coverage and connectivity reduce to compute the rank of a matrix
- Localisation of hole: reduces to the computation of a basis of a vector matrix, obtained by matrix reduction (as in Gauss algorithm).

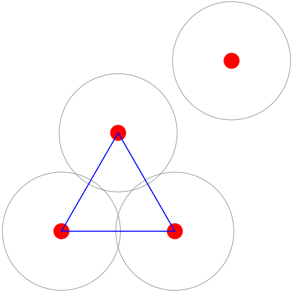
Cech complex



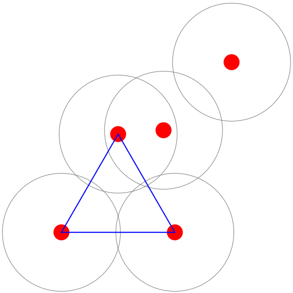
Cech complex



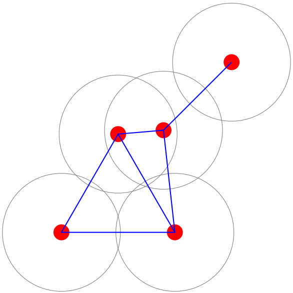
Cech complex



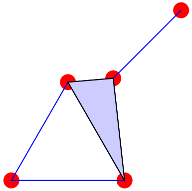
Cech complex



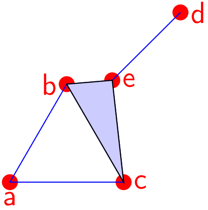
Cech complex



Cech complex



Cech complex



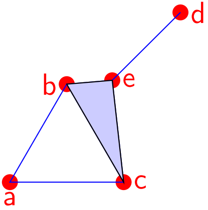
Vertices : a, b, c, d, e

Edges : ab, bc, ca, be, ec, ed

Triangles : bec

Tetrahedron : $\emptyset$

Cech complex



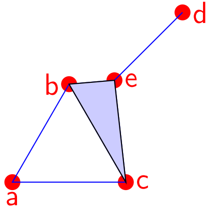
Vertices : a, b, c, d, e

Edges : ab, bc, ca, be, ec, ed

Triangles : bec

Tetrahedron : $\emptyset$

Cech complex



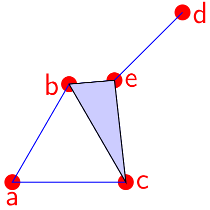
Vertices : a, b, c, d, e

Edges : ab, bc, ca, be, ec, ed

Triangles : bec

Tetrahedron : $\emptyset$

Cech complex



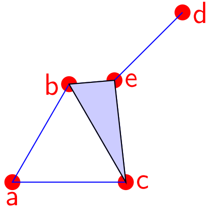
Vertices : a, b, c, d, e

Edges : ab, bc, ca, be, ec, ed

Triangles : bec

Tetrahedron : $\emptyset$

Cech complex



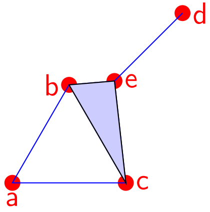
Vertices : a, b, c, d, e

Edges : ab, bc, ca, be, ec, ed

Triangles : bec

Tetrahedron : $\emptyset$

Cech complex



Vertices : $\{$ a, b, c, d, e $\} = \mathcal{C}_0$

Edges : $\{$ab, bc, ca, be, ec, ed $\} = \mathcal{C}_1$

Triangles : $\{$bec$\} = \mathcal{C}_2$

Tetrahedron : $\emptyset = \mathcal{C}_3$

Comput. Rips

Hypergraphs

A simplicial complex = hypergraph = boolean monotone function

Hypergraphs

A simplicial complex = hypergraph = boolean monotone function

The Embedded Homology of Hypergraphs and Applications
Stephane Bressan, Shiquan Ren, Jie Wu
arXiv:1610.00890

Cech complex

k-simplices

$$C_k = \bigcup \{ [x_0, \cdots, x_{k-1}],\ x_i \in \omega, \cap_{i=0}^{k} B(x_i, \epsilon) \neq \emptyset \}$$

Nerve theorem

We can read some topological properties of $\bigcup_{x \in \omega} B(x, \epsilon)$ on $(C_k,\ k \geq 0)$

- Same nb of connected components
- Same nb of holes
- Same Euler characteristic

Boundary operator

Definition

$$\partial_k \ : \ C_k \longrightarrow C_{k-1}$$
$$[v_0, \cdots, v_{k-1}] \longmapsto \sum_{j=0}^{k} (-1)^j [v_0, \cdots, \hat{v}_j, \cdots]$$

Boundary operator

Definition

$$\partial_k \; : \; C_k \longrightarrow C_{k-1}$$
$$[v_0, \cdots, v_{k-1}] \longmapsto \sum_{j=0}^{k} (-1)^j [v_0, \cdots, \hat{v}_j, \cdots]$$

Example

$\partial_2(bec) = ec - bc + be$

Boundary operator

Definition

$$\partial_k \;:\; C_k \longrightarrow C_{k-1}$$

$$[v_0, \cdots, v_{k-1}] \longmapsto \sum_{j=0}^{k} (-1)^j [v_0, \cdots, \hat{v}_j, \cdots]$$

Example

$\partial_2(bec) = ec - bc + be$

$\partial_1\partial_2(bec) = c - e - (c - b) + e - b = 0$

Theorem

$$\partial_k \circ \partial_{k+1} = 0$$

Consequence

$$\mathrm{Im}\ \partial_{k+1} \subset \ker \partial_k$$

Definition

$$H_k = \ker\ \partial_k / \mathrm{Im}\partial_{k+1} \text{ and } \beta_k = \dim\ \ker\ \partial_k - \mathrm{range}\ \partial_{k+1}$$

Interpretation : The magic

- β_0 : number of connected components
- β_1 : number of holes
- β_2 : number of voids
- to be continued

Example

$$\partial_0 \equiv 0,\ \partial_1 = \begin{pmatrix} -1 & 0 & 1 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 & 0 \end{pmatrix}$$

Nb of connected components

dim ker $\partial_0 = 5$, range $\partial_1 = 4$ hence $\beta_0 = 1$

Number of holes

$$\partial_2 = \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}$$

Nb of holes

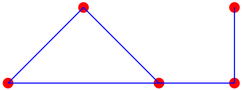
dim $\ker\partial_1 = 2$, range $\partial_2 = 1$ hence $\beta_1 = 1$

Polygons=cycles

$$\beta_1 = \text{ Nb of independent polygons} - \text{ Nb of independent triangles.}$$

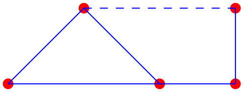
Polygons=cycles

$$\beta_1 = \text{ Nb of independent polygons} - \text{ Nb of independent triangles.}$$



Polygons=cycles

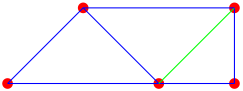
$$\beta_1 = \text{ Nb of independent polygons} - \text{ Nb of independent triangles.}$$



$$\beta_1 = 2 - 1 = 1.$$

Polygons=cycles

$$\beta_1 = \text{ Nb of independent polygons} - \text{ Nb of independent triangles.}$$



$$\beta_1 = 2 - 2 = 0.$$

Open question

What is the interpretation of the Betti numbers for hypergraphs or boolean monotone functions ?

Open question

What is the interpretation of the Betti numbers for hypergraphs or boolean monotone functions ?

Find the *single minimal triangulation* = construct the minimum weight basis of H_2

Euler characteristic ($S - A + F$)

Definition

$$\chi = \sum_{j=0}^{d} (-1)^j \beta_j$$

Discrete Morse inequality

$$-|\mathcal{C}_{k-1}| + |\mathcal{C}_k| - |\mathcal{C}_{k+1}| \leq \beta_k \leq |\mathcal{C}_k|$$

Euler characteristic $(S - A + F)$

Definition

$$\chi = \sum_{j=0}^{d} (-1)^j \beta_j = \sum_{j=0}^{\infty} (-1)^j \left|\mathcal{C}_j\right|$$

Discrete Morse inequality

$$-\left|\mathcal{C}_{k-1}\right| + \left|\mathcal{C}_k\right| - \left|\mathcal{C}_{k+1}\right| \leq \beta_k \leq \left|\mathcal{C}_k\right|$$

Alternative complex

Cech complex

$$[v_0, \cdots, v_{k-1}] \in \mathcal{C}_k \iff \cap_{j=0}^{k} B(x_j, \epsilon) \neq \emptyset$$

Rips-Vietoris complex

$$[v_0, \cdots, v_{k-1}] \in \mathcal{R}_k \iff B(x_j, \epsilon) \cap B(x_l, \epsilon) \neq \emptyset$$

k simplex $=$ clique of $k+1$ points

Difference RV vs Cech

For the l^∞ distance

RV=Cech

Euclidean norm : false negative

Rips complex may miss some holes

Difference RV vs Cech

For the l^∞ distance

RV=Cech

Euclidean norm : false negative

Rips complex may miss some holes

Cech vs Rips

$$\mathcal{R}_{\epsilon'}(\mathcal{V}) \subset \check{\mathrm{C}}_{\epsilon}(\mathcal{V}) \subset \mathcal{R}_{2\epsilon}(\mathcal{V}) \qquad \text{whenever} \qquad \frac{\epsilon}{\epsilon'} \geq \sqrt{\frac{d}{2(d+1)}}$$

Euclidean distance (D.-Feng-Martins)

- Coverage radius R_S
- Communication radius $R_C = \gamma R_S$

Lower-bound of the error

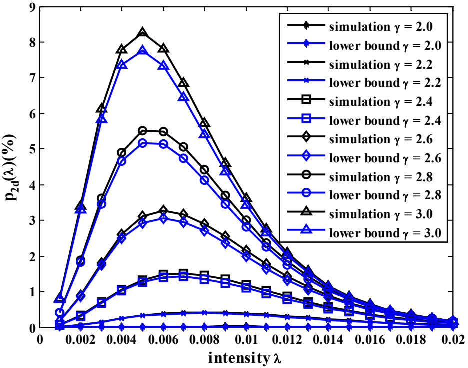
Theorem ($\sqrt{3} \leq \gamma \leq 2$)

$$p_{2dl}(\lambda) = 2\pi\lambda^2 \int_{R_s}^{R_c/\sqrt{3}} r_0\, dr_0 \int_{\varphi_l(r_0)}^{\varphi_u(r_0)} d\varphi_1 \int_{r_0}^{R_1(r_0,\varphi_1)} e^{-\lambda\pi r_0^2} \times e^{-\lambda|S^+(r_0,\varphi_1)|}\left(1 - e^{-\lambda|S^-(r_0,r_1,\varphi_1)|}\right) r_1\, dr_1 \tag{1}$$

where

$$\varphi_l(r_0) = 2\arccos(R_c/(2r_0)),\ \varphi_u(r_0) = 2\arcsin(R_c/(2r_0)) - 2\arccos(R_c/(2r_0))$$

$$R_1(r_0,\varphi_1) = \min\Big(\sqrt{R_c^2 - r_0^2\sin^2\varphi_1} - r_0\cos\varphi_1 \quad \sqrt{R_c^2 - r_0^2\sin^2(\varphi_1+\varphi_l(r_0))} + r_0\cos(\varphi_1+\varphi_l(r_0))\Big)$$



Probability to miss a hole using $\mathcal{R}_{R_S}$ and $\mathcal{R}_{R_C}$

Goals and related works

- Evaluate Betti nb and Euler charac. in some random settings
- Penrose : Asymptotics of $\mathbf{E}[|\mathcal{C}_k|^m]$ for Euclidian-RG Rips complex on the whole space ($m = 1, 2$)
- Kähle : Asymptotics of $\mathbf{E}[\beta_k]$ for Euclidian-RG Cech complex (deterministic number of points) and ER

Goals and related works

- Evaluate Betti nb and Euler charac. in some random settings
- Penrose : Asymptotics of $\mathbf{E}[|\mathcal{C}_k|^m]$ for Euclidian-RG Rips complex on the whole space ($m = 1, 2$)
- Kähle : Asymptotics of $\mathbf{E}[\beta_k]$ for Euclidian-RG Cech complex (deterministic number of points) and ER

Our results

Exact expressions of all moments of $|\mathcal{C}_k|$ and χ in any dimension for RG complex on a torus for the l^∞ norm

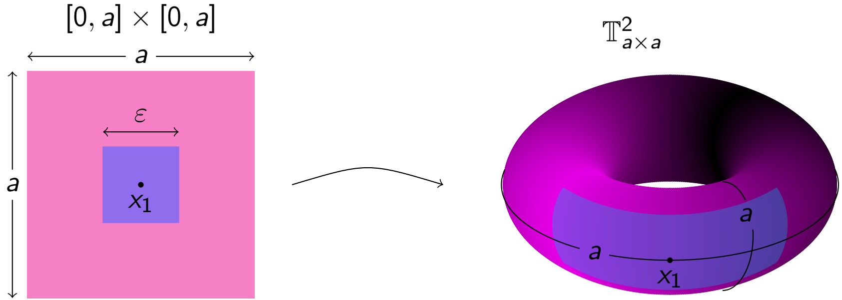
Outline

Algebraic topology

Poisson homologies
- Euler characteristic
- Asymptotic results
- Robust estimate

Persistence

Random setting



Euler characteristic

- d=1 : $\{\chi = 0 \cap \beta_0 \neq 0\} \Leftrightarrow$ { circle is covered }
- d=2 : $\{\chi = 0 \cap \beta_0 \neq \beta_1\} \Leftrightarrow$ { domain is covered }
- d=3 : $\{\chi = 0 \cap \beta_0 + \beta_2 \neq \beta_1\} \Leftrightarrow$ { space is covered }

Euler characteristic (D.-Ferraz-Randriam-Vergne)

Euler characteristic

$$\mathbf{E}\left[\chi\right] = -\frac{\lambda e^{-\theta\, a^d}}{\theta} B_d(-\theta\, a^d) \text{ where } \theta = \lambda \left(\frac{2\epsilon}{a}\right)^d.$$

where B_d is the d-th Bell polynomial

$$B_d(x) = \left\{ d \atop 1 \right\} x + \left\{ d \atop 2 \right\} x^2 + \ldots + \left\{ d \atop d \right\} x^d$$

k simplices

The key remark

$$|\mathcal{C}_k| = \int h(x_1, \cdots, x_k) d\omega^{(k)}(x_1, \cdots, x_k)$$

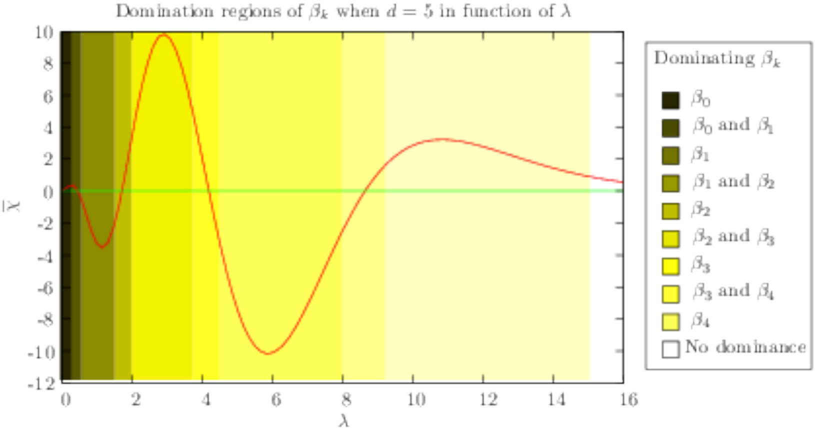
where

$$h(x_1, \cdots, x_k) \triangleq \frac{1}{k!} \prod_{i \neq j} \mathbf{1}_{\{\|x_i - x_j\| < \epsilon\}}$$

First moments

$$\mathbf{E}[|\mathcal{C}_k|] = \lambda a^d \, \frac{(k+1)^d}{(k+1)!} \, (a^d \theta)^k$$

Dimension 5


Domination regions of β_k when $d = 5$ in function of λ
$\overline{\chi}$
λ
10
8
6
4
2
0
-2
-4
-6
-8
-10
-12
0
2
4
6
8
10
12
14
16
Dominating β_k
β_0
β_0 and β_1
β_1
β_1 and β_2
β_2
β_2 and β_3
β_3
β_3 and β_4
β_4
No dominance

Asymptotic results

If $\lambda \to \infty$, $\beta_i(\omega) \xrightarrow{p.s.} \beta_i(\mathbb{T}^d) = \binom{d}{i}$.

Limit theorems

CLT for Euler characteristic

$$\text{distance}_{TV}\left(\frac{\chi - \mathbf{E}[\chi]}{\sqrt{V_\chi}},\ \mathfrak{N}(0,1)\right) \le \frac{c}{\sqrt{\lambda}}.$$

Limit theorems

CLT for Euler characteristic

$$\text{distance}_{TV}\left(\frac{\chi - \mathbf{E}[\chi]}{\sqrt{V_\chi}},\ \mathfrak{N}(0,1)\right) \le \frac{c}{\sqrt{\lambda}}.$$

Method

- Stein method
- Malliavin calculus for Poisson process

Concentration inequality

- Discrete gradient $D_x F(\omega) = F(\omega \cup \{x\}) - F(\omega)$
- $D_x \beta_0 \in \{1, 0, -1, -2, -3\}$

Concentration inequality

- Discrete gradient $D_x F(\omega) = F(\omega \cup \{x\}) - F(\omega)$
- $D_x \beta_0 \in \{1, 0, -1, -2, -3\}$

Concentration inequality

- Discrete gradient $D_x F(\omega) = F(\omega \cup \{x\}) - F(\omega)$
- $D_x \beta_0 \in \{1,\, 0,\, -1,\, -2,\, -3\}$

$c > \mathbf{E}[\beta_0]$

$$P(\beta_0 \geq c) \leq \exp\left[-\frac{c - \mathbf{E}[\beta_0]}{6} \log\left(1 + \frac{c - \mathbf{E}[\beta_0]}{3\lambda}\right)\right]$$

Complexity

An important remark

- Construction of the complex is exponential (worst case)

Complexity

An important remark

- Construction of the complex is exponential (worst case)
- Computations of Betti numbers is polynomial

Further application (D.-Martins-Vergne)

Green networking

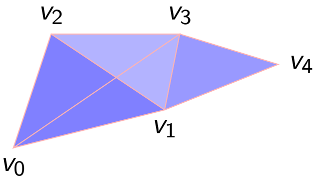
Switch off some sensors keeping the coverage

Height of an edge

Rank of the highest simplex it belongs to

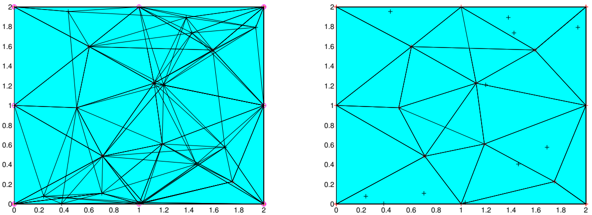
Index of a vertex

Infimum of the height of its adjacent edges



$$D[v_0, v_1, v_2] = D[v_0, v_1, v_3] = D[v_0, v_2, v_3] = D[v_1, v_2, v_3] = 3$$
$$D[v_1, v_3, v_4] = 2$$
$$I[v_0] = I[v_2] = 3 \text{ and } I[v_1] = I[v_3] = I[v_4] = 2$$

Example



- Complexity C bounded by 2^H

Complexity

$\theta_n = (r_n/a)^d$

$$\theta'_k = \frac{k^{\frac{1+\eta-d}{k-1}}}{n^{\frac{k}{k-1}}},\ \theta_k = \frac{k^{-\frac{1+\eta+d}{k-1}}}{n^{\frac{k}{k-1}}}$$

$$\theta_n \in [\theta'_k,\ \theta_k] \Longrightarrow C \xrightarrow{n\to\infty} k$$

Other regimes

Theorem (Critical: $n\theta_n \to 1$)

$C = O(n^3 \ln n)$.

Other regimes

Theorem (Critical: $n\theta_n \to 1$)

$C = O(n^3 \ln n).$

Theorem (Super-critiqual: $n\theta_n \to \infty$)

$$C_n = O(2^n n^3)$$

Outline

And then ?

Topological algebra

- Algebraic procedure to determine

And then ?

Topological algebra

- Algebraic procedure to determine
- β_0= nb of connected components

And then ?

Topological algebra

- Algebraic procedure to determine
- β_0= nb of connected components
- β_1= nb of holes

And then ?

Topological algebra

- Algebraic procedure to determine
- β_0= nb of connected components
- β_1= nb of holes

Problem

- Continuous structure (the cloud)

And then ?

Topological algebra

- Algebraic procedure to determine
- β_0= nb of connected components
- β_1= nb of holes

Problem

- Continuous structure (the cloud)
- Discrete result (Betti numbers)

And then ?

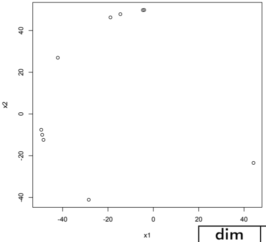
Topological algebra

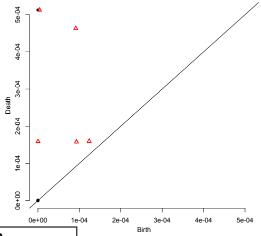
- Algebraic procedure to determine
- β_0= nb of connected components
- β_1= nb of holes

Problem

- Continuous structure (the cloud)
- Discrete result (Betti numbers)
- No continuity

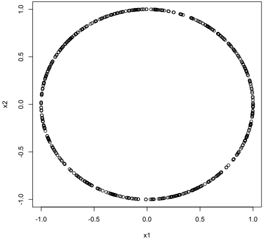
Persistence diagram of 10 pts on a circle

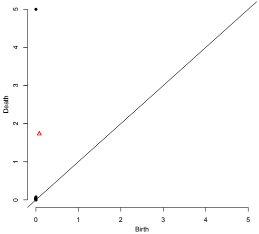




dim	Birth	Death
0	6.50655e-22	5.124479e-04
0	2.00255e-15	2.956910e-07
0	1.15748e-14	1.118440e-06
0	1.78540e-14	1.039640e-08
0	1.18076e-13	9.427080e-13
0	4.58441e-09	3.829860e-08
0	2.99134e-08	3.758330e-08
1	3.37283e-07	1.587770e-04
1	3.80028e-06	5.124480e-04
1	9.12595e-05	4.632000e-04
1	9.31748e-05	1.577760e-04
1	1.23796e-04	1.596950e-04

Persistence diagram of 500 pts on a circle



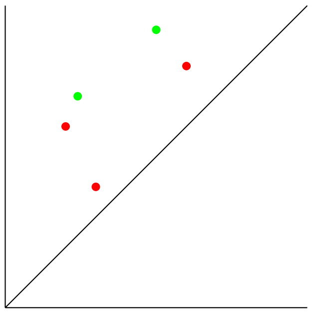


Comparison of persistence diagrams

Definition

If $|D_1| > |D_2|$, $\tilde{D}_1 = D_1 \setminus \{$ the $|D_1| - |D_2|$ pts of D_1 closest to the diagonal $\}$

$$\rho(D_1, D_2) = \mathfrak{T}_\infty(\tilde{D}_1, D_2)$$

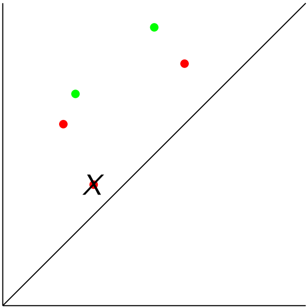


Comparison of persistence diagrams

Definition

If $|D_1| > |D_2|$, $\tilde{D}_1 = D_1 \setminus \{$ the $|D_1| - |D_2|$ pts of D_1 closest to the diagonal $\}$

$$\rho(D_1, D_2) = \mathfrak{T}_\infty(\tilde{D}_1, D_2)$$

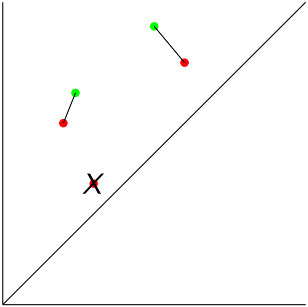


Comparison of persistence diagrams

Definition

If $|D_1| > |D_2|$, $\tilde{D}_1 = D_1 \setminus \{$ the $|D_1| - |D_2|$ pts of D_1 closest to the diagonal $\}$

$$\rho(D_1, D_2) = \mathfrak{T}_\infty(\tilde{D}_1, D_2)$$



Costs on configuration space

Definition (Total variation)

$$\mathfrak{C}_{\mathrm{TV}}(\omega, \eta) = \sup_{A \text{ compact}} |\omega(A) - \eta(A)| = (\text{ nb of} \neq \text{pts})$$

where

$$\omega = \sum_{j=1}^{n} \varepsilon_{x_i},\ \eta = \sum_{k=1}^{m} \varepsilon_{y_k}$$

Costs on configuration space

Definition (Total variation)

$$\mathfrak{C}_{\mathrm{TV}}(\omega,\,\eta) = \sup_{A\text{ compact}} |\omega(A) - \eta(A)| = (\text{ nb of} \neq \text{pts})$$

where

$$\omega = \sum_{j=1}^{n} \varepsilon_{x_i},\ \ \eta = \sum_{k=1}^{m} \varepsilon_{y_k}$$

Definition (Quadratic cost)

$$\mathfrak{C}_2(\omega,\,\eta) = \frac{1}{2}\inf\left\{ \int d_E(x,y)^2 d\beta(x,y),\ \beta \in \Sigma_{\omega,\eta} \right\},$$

$$= \begin{cases} +\infty & \text{if } m \neq n \\ \inf\ \frac{1}{2}\sum_{j=1}^{n} d_E(x_j, y_{\sigma(j)})^2 & \text{if } m = n < +\infty. \end{cases}$$

Finite point processes on $E = \mathbf{R}^d$

Theorem (LD'08)

$\mathfrak{T}_{\mathfrak{c}_2}(\mu, \nu) < \infty$ iff

$$\mu(\eta(E) = n) = \nu(\omega(E) = n), \ \forall n \geq 0$$
$$\sum_{n \geq 1} \mathcal{T}_{\mathfrak{c}_e}(j_n^{\mu}, j_n^{\nu})^2 \ \mu(\eta(E) = n) < +\infty$$

Moreover, the optimal map T is described by

$$T \ : \ \Gamma_E^{(n)} \longrightarrow \Gamma_E^{(n)}$$
$$\omega = \sum_{j=1}^{n} \varepsilon_{x_i} \longmapsto t_{j_n^{\mu}, j_n^{\nu}}(x_1, \cdots, x_n)$$

Poisson process

Theorem

- $\mathfrak{T}_e(\sigma_1, \sigma_2) < +\infty$
- t_{σ_1,σ_2} *the transport map of MKP*$(\sigma_1, \sigma_2, \mathfrak{C}_e)$
- *Then*

$$\begin{aligned} T \ : \ \chi_E &\longrightarrow \chi_E \\ \sum_{x\in\omega} \epsilon_x &\longmapsto \sum_{x\in\omega} \epsilon_{t_{\sigma_1,\sigma_2}(x)} \end{aligned}$$

is the transport map from π_{σ_1} *to* π_{σ_2} *and*

$$\mathfrak{T}_{\mathfrak{C}_2}(\pi_{\sigma_1}, \pi_{\sigma_2}) = \sigma_1(E)\, \mathfrak{T}_{\mathfrak{C}_e}(\sigma_1, \sigma_2).$$

Persistence diagrams of point processes

Theorem (LD, A. Vasseur'15)

- μ *and* ν *2 point processes*
- $\mathcal{D}^{\#}\mu$ = *distribution of the* μ*-persistence diagram*

$$\mathfrak{T}_\rho(\mathcal{D}^{\#}\mu,\, \mathcal{D}^{\#}\nu) \le \mathfrak{T}_{\mathfrak{c}_2}(\mu, \nu)$$